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Functional Itô calculus, path-dependence and the computation of Greeks

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Abstract

Dupire's functional Itô calculus provides an alternative approach to the classical Malliavin calculus for the computation of sensitivities, also called Greeks, of path-dependent derivatives prices. In this paper, we introduce a measure of path-dependence of functionals within the functional Itô calculus framework. Namely, we consider the Lie bracket of the space and time functional derivatives, which we use to classify functionals accordingly to their degree of path-dependence. We then revisit the problem of efficient numerical computation of Greeks for path-dependent derivatives using integration by parts techniques. Special attention is paid to path-dependent functionals with zero Lie bracket, called locally weakly path-dependent functionals in our classification. Hence, we derive the weighted-expectation formulas for their Greeks. In the more general case of fully path-dependent functionals, we show that, equipped with the functional Itô calculus, we are able to analyze the effect of the Lie bracket on the computation of Greeks. Moreover, we are also able to consider the more general dynamics of path-dependent volatility. These were not achieved using Malliavin calculus.

Keywords: Functional Itô calculus; Path-dependence; Greeks; Monte Carlo methods

1. Introduction

The theory of functional Itô calculus, introduced in Dupire's seminal paper [6], extends Itô's stochastic calculus to functionals of the current history of a given process, and hence provides an

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excellent tool to study path-dependence. Further work extending this theory and its applications can be found in the partial list [4,2,3,7–9,25].

We intuitively understand path-dependence of a functional as a measurement of its changes when the history of the underlying path varies. Here we propose a measure of path-dependence given by the Lie bracket of the space and time functional derivatives. Roughly, this is an instantaneous measure of path-dependence, since we consider only path perturbations at the current time. We then classify functionals based on this measure. Moreover, we analyze the relation of what we called *locally weakly path-dependent* functionals and the Monte Carlo computation of Greeks in path-dependent volatility models, cf. [13].

Malliavin calculus was successfully applied to derive these Monte Carlo procedures to compute Greeks of path-dependent derivatives in local volatility models, see for example [13,12,22, 15,14,23]. However, the theory presented here allows us to extend these Monte Carlo procedures to a wider class of path-dependent derivatives provided that the path-dependence is not too severe. This will be made precise in Section 3. We will also see that the functional Itô calculus can be used to derive the same weighted-expectation formulas shown in [13].

Furthermore, unlike the Malliavin calculus approach, we are also able to provide a formula for the Delta of functionals with more severe path-dependence, here called *strongly path-dependent*. In its current form, this formula enhances the understanding of the weights for different cases of path-dependence, although it is not as computationally appealing as the ones derived for locally weakly path-dependent functionals. It shows however the impact that the Lie bracket has on the Delta of a derivative contract. Additionally, the functional Itô calculus allows us to consider the more general path-dependent volatility models, see [10,16,17], for example.

Our main contribution is the introduction of a measure of path-dependence and the application of such measure to the computation of Greeks for path-dependent derivatives.

The paper is organized as follows. In Section 2, we provide some background on functional Itô calculus. Section 3 introduces the measure of path-dependence and a classification of functionals accordingly to this measure. Finally, we present applications of this measure of path-dependence to the computation of Greeks in Section 4. Two numerical examples, related to Asian options and quadratic variation contracts, are discussed.

2. A primer on functional Itô calculus

In this section we will present some definitions and results of the functional Itô calculus that will be necessary in Sections 3 and 4.

The space of $\mathbb{R}$-valued càdlàg paths in $[0, t]$ will be denoted by Λ_t. We also fix a time horizon $T > 0$. The *space of paths* is then defined as

$$\Lambda = \bigcup_{t\in[0,T]} \Lambda_t.$$

A very important remark on the notation: as in [6], we will denote elements of Λ by upper case letters and often the final time of its domain will be subscripted, e.g. $Y \in \Lambda_t \subset \Lambda$ will be denoted by Y_t. Note that, for any $Y \in \Lambda$, there exists only one t such that $Y \in \Lambda_t$. The value of Y_t at a specific time will be denoted by lower case letter: $y_s = Y_t(s)$, for any $s \leq t$. Moreover, if a path Y_t is fixed, the path Y_s, for $s \leq t$, will denote the restriction of the path Y_t to the interval $[0, s]$.

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