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Coupling and exponential ergodicity for stochastic differential equations driven by Lévy processes

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Abstract

We present a novel idea for a coupling of solutions of stochastic differential equations driven by Lévy noise, inspired by some results from the optimal transportation theory. Then we use this coupling to obtain exponential contractivity of the semigroups associated with these solutions with respect to an appropriately chosen Kantorovich distance. As a corollary, we obtain exponential convergence rates in the total variation and standard L^1-Wasserstein distances.

MSC: 60G51; 60H10

Keywords: Stochastic differential equations; Lévy processes; Exponential ergodicity; Couplings; Wasserstein distances

1. Introduction

We consider stochastic differential equations of the form

$$dX_t = b(X_t)dt + dL_t, \tag{1.1}$$

where $(L_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Lévy process and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field satisfying a one-sided Lipschitz condition, i.e., there exists a constant $C_L > 0$ such that for all x,

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$y \in \mathbb{R}^d$ we have

$$\langle b(x) - b(y), x - y \rangle \leq C_L |x - y|^2. \tag{1.2}$$

These assumptions are sufficient in order for (1.1) to have a unique strong solution (see Theorem 2 in [6]). For any $t \geq 0$, denote the distribution of the random variable L_t by μ_t. Its Fourier transform $\widehat{\mu}_t$ is of the form

$$\widehat{\mu}_t(z) = e^{t\psi(z)}, \quad z \in \mathbb{R}^d,$$

where the *Lévy symbol* (or *Lévy exponent*) $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by the Lévy–Khintchine formula (see e.g. [1] or [20]),

$$\psi(z) = i\langle l, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{\{|x|\leq 1\}})\nu(dx),$$

for $z \in \mathbb{R}^d$. Here l is a vector in $\mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

We call (l, A, ν) the *generating triplet* of the Lévy process $(L_t)_{t\geq 0}$, whereas A and ν are called, respectively, the *Gaussian covariance matrix* and the *Lévy measure* (or *jump measure*) of $(L_t)_{t\geq 0}$.

In this paper we will be working with pure jump Lévy processes. We assume that in the generating triplet of $(L_t)_{t\geq 0}$ we have $l = 0$ and $A = 0$. By the Lévy–Itô decomposition we know that there exists a Poisson random measure N associated with $(L_t)_{t\geq 0}$ in such a way that

$$L_t = \int_0^t \int_{\{|v|>1\}} vN(ds, dv) + \int_0^t \int_{\{|v|\leq 1\}} v\widetilde{N}(ds, dv), \tag{1.3}$$

where

$$\widetilde{N}(ds, dv) = N(ds, dv) - ds\, \nu(dv)$$

is the compensated Poisson random measure.

We will be considering the class of Kantorovich (L^1-Wasserstein) distances. For $p \geq 1$, we can define the *L^p-Wasserstein distance* between two probability measures μ_1 and μ_2 on $\mathbb{R}^d$ by the formula

$$W_p(\mu_1, \mu_2) := \left(\inf_{\pi \in \Pi(\mu_1,\mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x, y)^p \pi(dx\, dy) \right)^{\frac{1}{p}},$$

where ρ is a metric on $\mathbb{R}^d$ and $\Pi(\mu_1, \mu_2)$ is the family of all couplings of μ_1 and μ_2, i.e., $\pi \in \Pi(\mu_1, \mu_2)$ if and only if π is a measure on $\mathbb{R}^{2d}$ having μ_1 and μ_2 as its marginals. We will be interested in the particular case of $p = 1$ and the distance ρ being given by a concave function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(x) > 0$ for $x > 0$ as

$$\rho(x, y) := f(|x - y|) \quad \text{for all } x, y \in \mathbb{R}^d.$$

We will denote the L^1-Wasserstein distance associated with a function f by W_f. The most well-known examples are given by $f(x) = \mathbf{1}_{(0,\infty)}(x)$, which leads to the total variation distance