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Cartan, Schouten and the search for connection

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Abstract

In this paper we provide an analysis, both historical and mathematical, of two joint papers on the theory of connections by Élie Cartan and Jan Arnoldus Schouten that were published in 1926. These papers were the result of a fertile collaboration between the two eminent geometers that flourished in the two-year period 1925–1926. We describe the birth and the development of their scientific relationship especially in the light of unpublished sources that, on the one hand, offer valuable insight into their common research interests and, on the other hand, provide a vivid picture of Cartan's and Schouten's different technical choices. While the first part of this work is preeminently of a historical character, the second part offers a modern mathematical treatment of some contents of the two contributions.

MSC: 01A60; 53B20; 53B05

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1. Introduction

In 1926 the Dutch Academy of Sciences in Amsterdam published two short memoirs (Cartan and Schouten, 1926a and Cartan and Schouten, 1926b) on the theory of connections. They were joint papers by Élie Cartan and Jan Arnoldus Schouten, two of the most eminent geometers of the first half of the last century. The papers dealt with distinct but also closely related issues.

The first one introduced three different connections (now known as Cartan connections) on so-called group manifolds (in German, Schouten referred to them as *Gruppenmannigfaltigkeit*), i.e. on Lie groups. Two of these connections define an absolute parallelism of vectors, that is a parallelism that, contrary to the notion introduced by Levi-Civita in 1917, is independent of the path chosen to connect any two points of the manifold. By comparison, the third connection corresponds to a parallelism that is bounded to the path

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chosen and in the case of simple and semi-simple Lie groups coincides with the Levi-Civita parallelism of the underlying (pseudo-)Riemannian structure.

The second paper approached a difficult and most fascinating classification problem[1] consisting in determining all Riemannian manifolds that admit an absolute parallelism (i.e. a flat connection) that is consistent with the canonical (Levi-Civita) parallelism induced by the metric tensor.

In many respects these two papers stood out among other contributions to the theory of connections of the same period. In the first place, the collaboration with Schouten represented a noteworthy exception to the relative isolation of Cartan's research that had been following a peculiar and most original path since the early 1900's. In the second place, Cartan's and Schouten's joint works inaugurated in a certain way a new approach to the theory of Lie groups. The latter became to be regarded as abstract manifolds, thus allowing the emergence of a geometrical interpretation of classical results of Lie's theory. Eventually, the problems tackled in these papers greatly stimulated further geometrical investigations, especially in the realm of the theory of symmetric spaces, a theory that Cartan was led to develop precisely in the course of his collaboration with Schouten.

In view of these reasons, it seems useful to provide an analysis, both historical and mathematical, of these two papers. Our analysis will be greatly enriched by extensive recourse to unpublished sources kept in the Archives of the Academy of Sciences in Paris (AASP) and in Schouten's Nachlass at the Amsterdam Mathematical Centrum (AMC). The manuscript material which is relevant for our discussion amounts to 43 letters (AASP) + 120 letters (comprehending drafts by Schouten, AMC). The period covered by the correspondence runs from March 1924 up to June 1946. In view of our purposes, we will concentrate exclusively upon letters ranging from March 1924 to the late 1926.[2]

These unpublished sources enable us to get an adequate understanding of how their collaboration originated, by offering in addition precious insight into the heuristics of the discovery process. Finally, they provide a vivid picture of Cartan's and Schouten's quite diverging methods by testifying to the difficulty and the embarrassment that each of the two authors sometimes experienced when trying to understand the techniques of the other.

The present paper is divided into two parts. The first one (Sections 2–4) is mainly of a historical character. There the scientific contributions of Cartan and Schouten prior to their collaboration are sketchily described. Special emphasis is paid both in underlining their common interest in the dawning theory of connections and in highlighting their different technical approach. Sections 3 and 4 offer an analysis of the contents of Cartan and Schouten (1926a) and Cartan and Schouten (1926b) respectively, in the light of their scientific correspondence, which is here for the first time partly published. The second part of the paper is devoted to a modern mathematical treatment of some outcomes of their collaboration, namely to canonical connections on Lie groups and skew-symmetric torsion tensors.

2. A common background but different techniques

After the publication of Einstein's fundamental papers on the foundation of General Relativity, the whole realm of differential geometry experienced a period of intense and widespread development. A first manifestation of this phenomenon took place in 1917 when Tullio Levi-Civita published a memoir (Levi-Civita, 1917) in which he provided a geometrical interpretation of Christoffel symbols in terms of the notion of parallelism of vectors. From that moment on, the interest of geometers was devoted to investigating possible extensions of the notion of parallelism to manifolds of a more general type than those of a Riemannian kind. Accordingly, the local character of the space ceased to be exclusively Euclidean allowing the case of

[1] A precise formulation of the problem is given in Section 4 of this paper. In this respect, see also Wolf (1972).

[2] It might be useful to recall that an online database has recently been made available to study Cartan's *Nachlass*, his correspondence not being included. See the website http://eliecartanpapers.ahp-numerique.fr/.

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