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Heuristic analogy in *Ars Conjectandi*: From Archimedes' *De Circuli Dimensione* to Bernoulli's theorem

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A R T I C L E I N F O

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A B S T R A C T

This article investigates the way in which Jacob Bernoulli proved the main mathematical theorem that undergirds his art of conjecturing—the theorem that founded, historically, the field of mathematical probability. It aims to contribute a perspective into the question of problem-solving methods in mathematics while also contributing to the comprehension of the historical development of mathematical probability. It argues that Bernoulli proved his theorem by a process of mathematical experimentation in which the central heuristic strategy was analogy. In this context, the analogy functioned as an experimental hypothesis. The article expounds, first, Bernoulli's reasoning for proving his theorem, describing it as a process of experimentation in which hypothesis-making is crucial. Next, it investigates the analogy between his reasoning and Archimedes' approximation of the value of π, by clarifying both Archimedes' own experimental approach to the said approximation and its heuristic influence on Bernoulli's problem-solving strategy. The discussion includes some general considerations about analogy as a heuristic technique to make experimental hypotheses in mathematics.

By which methods do mathematicians solve problems? This question has received considerable philosophical attention in the last two decades in the context of increased research into the philosophy of mathematical practice (see, for instance, Cellucci, 2002; 2017, Grosholz and Breger 2000; Mancosu 2008; Ferreirós, 2015). The question is often addressed by careful logical and philosophical analysis of historical case studies (e.g. Gillies, 1995; Grosholz, 2000; Mancosu, 1999). The result is better philosophical understanding of both mathematical methods and the historical development of specific areas of mathematics. This article investigates an important case in the history of mathematical probability. It aims to contribute a perspective into the question of problem-solving methods in mathematics while also contributing to the comprehension of the historical development of mathematical probability. Specifically, the article investigates the way in which Jacob Bernoulli proved the main mathematical theorem that undergirds his art of conjecturing—the theorem that founded, historically, the field of mathematical probability (Bernoulli, 2006). Bernoulli created an analogy between an urn filled with pebbles to the atmosphere, on the one hand, and to the human body, on the other, as an argument to warrant the applicability of the mathematics of probability to natural and social events (Daston, 1988, pp. 230-253). Meanwhile, his theorem provided the mathematical basis upon which conjectures about such events could be made scientifically (Sylla, 1998). I will argue that Bernoulli proved his theorem by a process of mathematical experimentation in which the central heuristic strategy was analogy. In this context, the analogy functioned as an experimental hypothesis.

I take as starting points the views (1) that mathematics is an experimental practice that investigates hypothetical states of things (Peirce 1976; Campos, 2007; 2010; Carter, 2014)[1] and (2) that mathematics is a heuristic practice and primarily consists in problem solving (Cellucci, 2002; 2017). Thus, by "experimental hypotheses" I mean the plausible solutions that mathematicians essay in their attempts to solve problems, usually within an existing mathematical framework. These hypotheses consist in experimental modifications to an existing "diagram"—be it a geometrical figure, an algebraic equation, and so on—in order to derive another "diagram" presenting the sought mathematical result. In this context, a "diagram" is an iconic presentation of mathematical

[1] Following standard practice in Peirce scholarship, references to Peirce 1976 will be abbreviated NEM followed by volume and page number, while references to Peirce, 1931-1958 will be abbreviated CP followed by volume and paragraph number.

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objects and relations that conform to a hypothetical state of things.[2] The experimental analogy that guides Bernoulli's reasoning is, in this case, between two mathematical problems. Bernoulli conceived of his own problem of approximating the unknown *a priori* probability of events on the basis of empirical observation as being analogous to Archimedes' problem of approximating the value of π in *De Circuli Dimensione* (Κύκλου μέτρησις in the original Greek). Given this perceived analogy between the problems, Bernoulli pursued a solution strategy analogous to that of Archimedes. This analogy, moreover, belongs within a context of mathematical experimentation in which a problem is posed and solutions are essayed experimentally.

In order to make this case, I will first expound Bernoulli's reasoning for proving his theorem, describing it as a process of experimentation in which hypothesis-making is crucial. I will then investigate the analogy with Archimedes' approximation of the value of π, by clarifying both Archimedes' own experimental approach to the said approximation and its heuristic influence on Bernoulli's problem-solving strategy. The course of this discussion will include some general considerations about analogy as a heuristic technique to make experimental hypotheses in mathematics.

1. Bernoulli's demonstration

In his *Ars Conjectandi* (1713), Bernoulli states and demonstrates the first law of large numbers—arguably the most important breakthrough of early mathematical probability theory. Bernoulli sought to find a way to estimate the probabilities of events on the basis of empirical observation in situations where the probabilities cannot be estimated *a priori*.[3] There has been ample discussion of the logical implications of Bernoulli's mathematical findings and of his arguments to warrant the application of mathematical probability theory, on the basis of his theorem, to problems involving natural and social events (see, for instance, Pearson, 1925; Sheynin, 1968; Hacking, 1971; Stigler, 1986, pp. 62-78, and Sylla, 1998). Here I will discuss the strictly mathematical problem that Bernoulli poses and resolves. It is, in fact, a two-fold problem. First, Bernoulli wants to show that as the number of empirical observations increases and tends towards infinity, there is no bound to the degree of accuracy with which the observed statistical frequency of an event approaches the *a priori* probability as its asymptote. He claims that by an "instinct of nature" we know that the more observations we have the lesser the risk involved in estimating the *a priori* probability from the *a posteriori* ratio. However, what we know by instinct requires mathematical demonstration (Bernoulli, 2006, p. 328). That is, Bernoulli *hypothetically poses*, on the basis of an instinct of nature, a mathematical proposition, and this proposition becomes a problem that he must resolve. Stephen Stigler observes that the "empirical approach to the determination of chances was not new with Bernoulli, nor did he consider it to be new. What was new was Bernoulli's attempt to give formal treatment to the vague notion that the greater the accumulation of evidence about the unknown proportion of cases, the closer we are to certain knowledge about that proportion" (Stigler, 1986, p. 65). In contemporary terms, Ian Hacking (1971) has suggested that Bernoulli approaches the problem formally by trying to show that as the number of empirical observations of a binomial event tends towards infinity, the probability that the difference between the *a priori* probability and the observed frequency is less than any small number approaches 1. In other words, Bernoulli seeks to prove the following hypothetical mathematical proposition:

> Let p be the probability of a successful event E being the outcome on any chance experimental trial. Let n be the number of experimental trials, x be the number of successes in n trials, and $s_n = x/n$ be the proportion of successes in n trials. Show that for any small positive number ε, the probability $P(|p - s_n| < \varepsilon) \to 1$ as $n \to \infty$.[4]

In this way, Bernoulli wants to justify mathematically that s_n is a good estimate of p.

Second, Bernoulli seeks to show that the required number of experimental trials may be specified mathematically in order to ensure that the empirical estimate is as close to the *a priori* probability of an event as is desired. Continuing with our foregoing notation, we may express Bernoulli's statement of the problem as follows:

Show that n may be specified such that, for any given large positive number c

$$P(|p - (x/n)| \leq \varepsilon) > c\, P(|p - (x/n)| > \varepsilon).$$ [5]

Stigler emphasizes that Bernoulli proved more than just the first law of large numbers. Had Bernoulli only considered the first part of the problem, it would be strictly fair to call his theorem just the first "weak law of large numbers." However, mainly on account of the second part, Bernoulli's "actual result was deeper, subtler, more precise, more difficult, and more ambitious than the simple and elementary statement of the weak law of large numbers" (Stigler, 1986, p. 66). Bernoulli demonstrated formally that, as the number of observations increases to infinity, the probability that the difference between the observed frequency and the *a priori* probability is arbitrarily small tends to 1. Moreover, he showed how to determine the number of observations required in order to attain a desired level of accuracy in the statistical estimate. With this in

[2] For this Peircean sense of a mathematical diagram or iconic sign see, for instance, NEM 4, p. 219 footnote, and my commentary in Campos, 2007 and 2010.

[3] Regarding Bernoulli's reasons to focus on these kinds of observational situations, rather than on games of chance in which probabilities can be estimated *a priori*, see Sylla's "Introduction" to Bernoulli, 2006. See also Collani (2006), who argues that by undertaking to develop an *ars conjectandi* Bernoulli intended to address the main philosophical problem of his era in Europe, namely, uncertainty. Collani surmises that Bernoulli, influenced by Pierre Bayle, was concerned with the religious dogmatism and opinionated certainty that caused political violence in Europe. Thus Bernoulli wanted to develop an epistemological alternative, a method to make more temperate or moderate decisions under uncertainty. Thus his motives were not primarily mathematical but scientific and epistemological. Other relevant references include Shafer, 1996 and Schneider, 2005.

[4] See Hacking, 1971, pp. 221-222, and Stigler, 1986, pp. 63-70.

[5] I adapt this expression from Stigler, 1986, p. 66. Stigler points out several possible mathematical and conceptual pitfalls related to stating Bernoulli's result in contemporary probabilistic terms. Among these, the most salient is that Bernoulli only treats the case where the numbers of successes, r, and failures, s, are integers, and not with the contemporary situation in which the ratio $p = r/(r+s)$ ranges over all the real numbers in the interval $[0, 1]$. For details, see p. 66-67. More importantly, an anonymous reviewer has pointed out several historical risks with using contemporary mathematical notation and probabilistic concepts to express Bernoulli's theorem and reasoning. First, for the concept represented in this expression by a capital P, Bernoulli uses the word "*verisimilis*" which Sylla accurately translates as "likely" rather than "probable" (Bernoulli, 2006, p. 337). Bernoulli's point is that the number of observations can be estimated in order for the empirical estimate to reach a desired verisimilitude from the limited epistemic perspective of the conjecturer. Second, Bernoulli does not use the Latin word "*probabilis*" to refer to the *a priori* ratio of fecund cases to all the cases. The contemporary mathematical expression inserts the use of probability as a frequency ratio into Bernoulli's reasoning, where it did not have that use. The lower-case p in this expression, then, may be thought to represent ratio or proportion (Latin *ratio*) rather than a frequentist probability. I agree with these observations and cite the contemporary expressions with caution in order to engage the existing literature while being mindful of the risks. This situation is hard to avoid, as exemplified by Eugene Seneta's history of the law of large numbers that begins with Bernoulli's theorem (2013).

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