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# Leibniz on the requisites of an exact arithmetical quadrature

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ABSTRACT

In this paper we will try to explain how Leibniz justified the idea of an exact arithmetical quadrature. We will do this by comparing Leibniz's exposition with that of John Wallis. In short, we will show that the idea of exactitude in matters of quadratures relies on two fundamental requisites that, according to Leibniz, the infinite series have, namely, that of regularity and that of completeness. In the first part of this paper, we will go deeper into three main features of Leibniz's method, that is: it is an infinitesimal method, it looks for an arithmetical quadrature and it proposes a result that is not approximate, but exact. After that, we will deal with the requisite of the regularity of the series, pointing out that, unlike the inductive method proposed by Wallis, Leibniz propounded some sort of intellectual recognition of what is invariant in the series. Finally, we will consider the requisite of completeness of the series. We will see that, although both Wallis and Leibniz introduced the supposition of completeness, the German thinker went beyond the English mathematician, since he recognized that it is not necessary to look for a *number* for the quadrature of the circle, given that we have a *series* that is equal to the area of that curvilinear figure.



## 1. Introduction

In the Parisian period (1672–1676), Leibniz developed a method for dealing with the problem of the quadrature of the circle and of conic sections, which has three main features: it is an infinitesimal method; it looks for an arithmetical quadrature; and it proposes a result that is not approximate, but exact. If we take into account the first two features, Leibniz's proposal is relatively similar to that presented by John Wallis in his *Arithmetica infinitorum*. However, the desired exactitude in matters of quadratures, especially in regards to the quadrature of the circle, is certainly one of Leibniz's main innovations. In this paper we will try to explain how Leibniz justified the idea of an exact quadrature. Given the two coincidences with Wallis' proposal that we have pointed out, we will do this by comparing Leibniz's exposition with that of the English mathematician. In short, we will show that the idea of exactitude in matters of quadratures relies on two fundamental requisites that, according to Leibniz, the infinite series have, namely, that of regularity and that of completeness. We will divide this paper into three parts. In the first one, we will go deeper into the features of Leibniz's method that we have just pointed out. In the second, we will deal with the requisite of the regularity of the series, pointing out that, unlike the inductive method proposed by Wallis, Leibniz propounded some sort of intellectual recognition of what is invariant in the series. Finally, in the third part, we will consider the requisite of completeness of the series. We will see that, although both Wallis and Leibniz introduced the supposition of completeness, the German thinker went beyond the English mathematician, since he recognized that it is not necessary to look for a *number* for the quadrature of the circle, given that we have a *series* that is equal to the area of that curvilinear figure.

## 2. Main features of Leibniz's method for quadratures

We will examine in more detail the three chief features of the method proposed by Leibniz for quadratures in the Parisian period. We based this description on the important treatise *De quadratura arithmetica circuli, ellipseos et hyperbolae, cujus corollarium est trigonometria sine tabulis*, which Leibniz wrote between 1675 and 1676 (from now on: DQA), and on other texts of the same period, in which he dealt with the same problem and which were preliminary for that treatise.

First, it is an infinitesimal method. Beyond the question about what is for Leibniz an infinitesimal (for this, see Arthur, 2009; Raffo Quintana, 2016, pp. 208–213), this implies that his procedure is inscribed in the line of the 'geometry of indivisibles', which at the same time is inserted in the tradition of Archimedes' geometry (on

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Leibniz's division of geometry, see Probst, 2012, pp. 149–150). Leibniz confesses his admiration for Archimedes, especially because he was the first to show the key of geometry, namely, "the Indivisibles, or, if you prefer, the infinitely small (…)" (A VII 6, 498).[1] Thus, when Leibniz refers to the method or to the geometry of indivisibles, he is not referring exclusively to the procedure presented by Bonaventura Cavalieri in his *Geometria indivisibilibus continuorum quadam nova ratione promota* of 1635. Leibniz points out that, "(…) when I speak of the Geometry of indivisibles, I envisage something much wider than that of Cavalieri, which seems to me to be a poor portion of that of Archimedes" (A VII 6, 498).[2] This explains the fact that Leibniz refers to his own infinitesimal method as a 'method of indivisibles', since it is inscribed in the tradition of geometry that starts with Archimedes, although he does not employ indivisibles (A VII 6, 521).

Second, as the title of the treatise suggests, Leibniz's method aims to achieve an *arithmetical* quadrature. In this sense, as we have said, he followed the path traced by John Wallis in his *Arithmetica infinitorum* of 1656, which was read by Leibniz in his years in Paris (Hofmann, 1973, pp. 247–251; Beeley, 2007, pp. 65–68). In the seventeenth century, the opinion that the problem of the quadrature of the circle, that is, the construction of a square equal in area to a circle, cannot be solved through the classical methods of geometry, was quite widespread (Jesseph, 1999, pp. 20–26; a more detailed account can be found in; Crippa, 2014). This does not mean that the treatment of this problem was abandoned, but that the new approaches were different from those of classical geometry. As Lützen points out, the mathematicians of the seventeenth century were not mainly concerned with the geometric constructability of the quadrature of the circle, but with the analytic expression of its area (2014, p. 246; see also Lützen's useful classification of the different quadratures of the circle in pp. 215–218). An example of one of these new approaches is Wallis' idea that arithmetic is the foundation of all mathematics, and thus the results of geometry, which is subordinated to arithmetic, can be obtained more clearly by appealing to arithmetic. In regards especially to the problem of the quadrature, this implies representing a sequence of approximations (such as those described in Archimedes' method) in an arithmetical way, that is, as infinite sums whose results can be obtained through an arithmetical calculus (Mancosu, 1996, pp. 86–87; Jesseph, 1999, pp. 37–38 and 173–188; Beeley, 2013). Similarly, Leibniz understands that there is another resource which is to 'curvilinear geometry' (that is, the kind of geometry which deals with problems that cannot be reduced to equations, such as, for example, the problem of the quadrature of the circle) what the algebraic equations are to 'rectilinear geometry' (which by contrast includes problems that can be reduced to equations), namely, the *progressions* or *series*, which pertain to arithmetic:

> Although the *Equations* abandon us at this time, the Nature of things provides us with other means, namely, *Progressions*. For, just as rectilinear Problems are reduced to the calculus and to numbers by means of Equations, in the same way the difficulty of Curvilinear [problems] is transferred from Geometry to Arithmetic by means of progressions. (A VII 6, 88–89)[3]

This appeal to the series is fundamental for Leibniz's proposal, since it allows him to transform a geometrical problem into an arithmetical one. The importance of this is expressed in the two goals which Leibniz recognized that his infinitesimal method has, namely, on the one hand, that it allows a circle or any other curvilinear figure to be transmuted into a equipollent rational figure, and, on the other hand, that it allows a circle or the considered curvilinear figure to be exhibited by means of infinite rational sums (A VII 6, 641).

Third, Leibniz proposes an *exact* quadrature, that is, which provides a value which is not merely approximate. Leibniz emphasizes this in the preface of the treatise DQA (A VII 6, 170–174). With the proposal of an exact quadrature of the circle, he goes even beyond Wallis, who showed how to express with the greatest proximity the quadrature of the circle through numbers, that is, to the extent that the nature of numbers allows it (Wallis, 1656, *Dedicatio*, sig. Bb2). Indeed, at the end of the *Arithmetica infinitorum*, Wallis confesses that the ratio of the circle to the square did not appear as clearly as he would have wished (Wallis, 1656, p. 197). One year later, in his *Mathesis universalis* (1657, p. 219), he wrote that those who believe they have found an exact value are delirious (see Jesseph, 1999, p. 12, note 20). However, as Crippa (2014, pp. 393–395) points out, after reading Mercator's *Logarithmotechnia* (1668), Wallis accepted the idea of an exact quadrature of the sector of an hyperbola, since there is a series equal to it, and Leibniz knew Wallis' review of Mercator's work. Now, although Leibniz was not the first in giving an exact arithmetical quadrature, one of his main contributions was to provide an exact arithmetical quadrature of the circle. Thus, it seems clear that, at least in Leibniz's view, Wallis was committed with the arithmetical quadrature of the hyperbola, but not of the circle.

By proposing an exact arithmetical quadrature of the circle, in some way Leibniz also challenges the boundaries that Descartes' *Géométrie* had proposed in matters of exactitude. According to Descartes, only the curves capable of being expressed through algebraic equations are considered in analytic geometry (Descartes, 1659, pp. 21–23). In Cartesian geometry, geometrical curves are distinguished from mechanical curves, which are not exact, like the first ones, since for their construction approximations to infinity are required. In this sense, for Descartes 'exactitude' and 'algebricity' are synonymous terms (Breger, 1986, p. 120; see also Knobloch, 2006, pp. 116–120; and Probst, 2012, pp. 151–152). Thus, the infinitesimal methods, which proceed by infinite approximations, are beyond the scope of analytic geometry. Many years later, in *Du nouveau système de l'infini* of 1703, Michel Rolle raised the same criticism regarding infinitesimal mathematics: "[b]ut it seems that this feature of exactness does not reign anymore in geometry since the new system of infinitely small quantities has been mixed to it" (Rolle, 1703, p. 312; translation from Mancosu, 1996, p. 165).[4] In proposing an exact quadrature of the circle, Leibniz believes he has overcome one of the limitations that the infinitesimal methods of that time had, namely, the fact that they always provided approximations, unlike the exactitude which the analytic geometry of Descartes implied. In this sense, we say that he wants to propose an infinitesimal method whose result is not approximate, but exact.

[1] "Indivisibilia certe, aut si mavis infinite parva (…)". Unless otherwise stated, translations are ours. We will refer to Leibniz, 1923 et seq. following the standard abbreviation: A, followed by series (in Roman numerals), volume (in Arabic numerals) and page number. Ex.: A VII 6, 600.

[2] "Porro cum de Geometria indivisibilium loquor longe aliquid Cavaleriana amplius intelligo, quae mihi non videtur esse nisi portio mediocris Archimedeae".

[3] "Mais quoyque les *Equations* nous abandonnent en cette rencontre, la Nature des choses n'a pas laissé pourtant de nous fournir un autre moyen, sçavoir les *Progressions*. Car comme les Problemes rectilignes, se reduisent au calcul et aux nombres par les Equations; de même la difficulté des Curvilignes est transferée de la Geometrie a l'Arithmetique par les progressions".

[4] "Mais il semble que ce caractere d'exactitude ne regne plus dans la Géométrie depuis que l'on y a mêlé le nouveau Systême des Infiniment petits".