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Riemann's and Helmholtz-Lie's problems of space from Weyl's relativistic perspective

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ABSTRACT

I reconstruct Riemann's and Helmholtz-Lie's problems of space, from some perspectives that allow for a fruitful comparison with Weyl.

In Part II. of his inaugural lecture, Riemann justifies that the infinitesimal metric is the square root of a quadratic form. Thanks to Finsler geometry, I clarify both the implicit and explicit hypotheses used for this justification. I explain that Riemann-Finsler's kind of method is also appropriate to deal with indefinite metrics. Nevertheless, Weyl shares with Helmholtz a strong commitment to the idea that the notion of group should be at the center of the foundations of geometry. Riemann missed this point, and that is why, according to Weyl, he dealt with the problem of space in a "too formal" way. As a consequence, to solve the problem of space, Weyl abandoned Riemann-Finsler's methods for group-theoretical ones.

However, from a philosophical point of view, I show that Weyl and Helmholtz are in strong opposition. The meditation on Riemann's inaugural lecture, and its clear methodological separation between the infinitesimal and the finite parts of the problem of space, must have been crucial for Weyl, while searching for strong epistemological foundations for the group-theoretical methods, avoiding Helmholtz's unjustified transition from the finite to the infinitesimal.

1. Introduction

The emergence of the two theories of relativity called for a reconstruction of *the problem of space* (abbreviated as **p.o.s.** in the following text). The p.o.s. consists in searching to justify the choice of the geometrical axioms that are adopted in order to describe physical space(-time). In the relativistic context, it is a matter of justifying the use of differential geometry, in particular of Riemannian and pseudo-Riemannian metrics. The best known relativistic constructions of the p.o.s. are those of Weyl and Cartan, but there are other constructions, like Becker-Blaschke's p.o.s., which seem to have been insufficiently emphasised by historians and philosophers.[1]

Not only did Weyl furnish us with one of the first relativistic solutions to the p.o.s., he also played an important role in making known the previous (non relativistic) positions regarding it, more specifically those of Riemann and Helmholtz-Lie. Indeed, while he was in the middle of his period of intensive work on space, Weyl was requested to prepare a commented edition of Riemann's habilitation conference. This resulted in three editions from 1919 to 1923,[2] which contributed to the view of Riemann as the founding father of the *mathematical* p.o.s. —in the precise sense of the search for a justification to the quadratic nature of the metric—, and as such the initiator of a new *philosophical* era in the p.o.s. Weyl used his knowledge about infinitesimal geometry in order to provide a better understanding of Helmholtz's solution to the p.o.s., which was interpreted as a requirement of maximal isotropy (or flag-isotropy, to use the current terminology) thanks to the reconstruction by Sophus Lie.[3] Weyl's opinions on the place of Riemann and of Helmholtz-Lie in the history of the p.o.s. are also included in the first conferences of *Mathematische Analyse des Raumproblems*.[4]

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[1] Cf. Bernard (2015a).

[2] Riemann (1919, 1921, 1923), the first edition being Riemann (1867).

[3] Weyl explains this reconstruction for example in (Weyl, 1949, p. 81).

[4] Using the new framework of purely infinitesimal geometry, (Weyl, 1923a, pp. 29–32) characterises Euclidean space by the integrability of the parallel transports of vectors, apparently suggested by (Riemann, 1898, p. 294). (Weyl, 1923a, pp. 32–39) then proves the Riemannian proposition according to which the possibility to move finite figures rigidly in space implies that space is either Euclidean, or a sphere, or a pseudo-sphere. (Weyl, 1923a, pp. 39–40) exposes Riemann's argument that provides justification for the fact that the metric is a positive definite quadratic form. Finally (Weyl, 1923a, pp. 40–58), deals with Helmholtz-Lie's p.o.s. See also Audureau and Bernard, 2015; Bernard, 2015b.

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Therefore, Weyl is not only the author of one of the main relativistic reconstructions of the p.o.s., but also an important “historian” of the p.o.s. What is meant here by “history”? Weyl did not care about historical fidelity in his presentation of Riemann's and Helmholtz-Lie's opinions. He rephrased both, mathematics and epistemological outlook of the authors he talked about, in order to use them within the construction of his new relativistic position. These past positions furnish technical tools (concepts, theorems, mathematical frameworks) that are partly rehabilitated in the relativistic p.o.s. in reinterpreted forms. They are also important in so far as Weyl criticizes them, in order to precise his own *philosophical* position on space.

In the present article, I aim at reconstructing Riemann's and Helmholtz-Lie's p.o.s. from some perspectives that allow for a fruitful comparison with Weyl's p.o.s. Because Riemann's and Helmholtz's texts are not of the same nature, and because Weyl's relationship to them is different, I will adopt different perspectives and methodologies in both parts of the article. As a general background, in reconstructing these p.o.s., I will always have in mind Weyl's comments on the p.o.s.

The first section (numbered 2.), concerning Riemann's p.o.s., will use more recent mathematics (Finsler geometry) in order to clarify some technical problems. Riemann's habilitation memoir is full of such problems –some of which are still puzzling us today–, and contains only a few explicit philosophical theses. Even if these theses were very sketchy in Riemann (1898), Weyl's epistemological position on space was deeply influenced by them. However, when Weyl turned to his relativistic p.o.s., he abandoned Riemann's and Finsler's types of method, and expressed his problem in terms of continuous groups of transformations. Concerning Weyl's problem of space and the role of the notion of group within it, the reader can also have a look at: Weyl, 1922; Weyl, 1923c; Eckes, 2011; Scheibe, 1957; Scheibe, 2001; Scholz, 1994; Scholz, 1995; Scholz, 1999, Scholz, 2012. My reconstruction of Riemann's p.o.s. will contribute to clarifying the reasons for this abandoning of Riemann-Finsler's methods.[5]

There is also another reason, from a Weylian point of view, to deal with the technical problems of the first part of Riemann's memoir. This regards the fact that a lot of commentators insisted that Riemann did not give a complete proof that the metric is the square root of a positive definite quadratic form. Instead, they claim that Riemann's text is obscure, and that he only gave some arguments to show that such a metric is a good possible choice *from among others*. The arguments given by Riemann are, in this sense, *incomplete*. Maybe, as I am inclined to believe, Riemann did not intend even to give such completely selective arguments. Nevertheless, this incompleteness is problematic from a Weylian point of view. Indeed, Weyl wanted to interpret this part of Riemann's memoir as a very attempt to give *a priori* foundations to the notion of metric. For Weyl, “*a priori*” has a strong philosophical meaning, referring to apodictic (i.e. universal and necessary) requirements. Through my work on the implicit hypotheses of Riemann, I try to complete Riemann's deduction in order to select uniquely the Riemannian class of metrics. At the end of section 2.3, I will effectively sketch a way to complete Riemann's selection of such a class.

In contrast with Riemann's text, Helmholtz's texts included less technical difficulties, at least after the clarification brought about by Lie in the 1870s and the subsequent developments of Lie groups and algebras. The philosophical parts of Helmholtz's texts from the period 1866–1870 are much more developed than in Riemann's text, and are mainly in opposition to Weyl's philosophy of space in the period 1916–1923, when it was a kind of transcendental idealism applied to the domain of the infinitesimal.[6] In his own texts, Weyl recognizes the importance of Helmholtz's construction for the pre-relativistic p.o.s. By adopting the framework of the continuous groups of transformations, Weyl's relativistic p.o.s. uses methods that are closer to Helmholtz-Lie than Riemann. However, the specific axioms used by Helmholtz, expressing isotropy and monodromy of space, must be abandoned in the relativistic context, even at the infinitesimal scale. Strangely enough, Weyl gives only *technical* reasons for abandoning them (relative to the anisotropy of Lorentzian metrics), but he does not comment at all on Helmholtz's philosophical construction.

The second section (numbered 3.) will thus avoid technicalities and get directly to the clarification of the philosophical Helmholtzian claims, and the reasons why Weyl was silent about them.

2. Riemann's p.o.s.

2.1. The “pluralisation” of geometry

The new era in the p.o.s., which began with Riemann, is the result of a gradual discovery (in the early decades of XIXth century) of the existence of many –even infinitely many– possible geometrical systems for geometry.[7] It was no longer possible, at least from a purely logical point of view, to speak about geometry in the singular. It was once and for all recognised as plural. Bolyai and Lobachevsky revealed to us the possibility of a new synthetic geometry, through a special way of denying Euclid's fifth postulate. Moreover, the geometrical study of curved lines and surfaces had flourished in the century that preceded Riemann's work. One can mention Euler and Gauss as two great figures of this early work on curved lines and surfaces. Unlike lines, surfaces are subject to a study of their intrinsic metric properties that allow us to discover an infinite number of possible forms. Nevertheless, Euler, Gauss (Gauss 1827), and their respective contemporaries, studied the intrinsic metrical properties of surfaces only through the assumption of a Euclidean ambient space in which the surfaces were embedded. In this sense, these early mathematics of curvature did not provide a theory of curved space, but rather a theory of the *embedding* of curved surfaces *in* ordinary flat space; even if it was then often a matter of focusing on the properties that were independent of the chosen embedding.

2.2. The epistemological structure of Riemann's memoir

Riemann brought intrinsic geometry to a new stage in its history, by breaking totally free of the supposition of an ambient space. Weyl published his conference as a three-part memoir, this plan proving to be decisive in the epistemological structure of the p.o.s.[8]

Indeed, in parts I.+II., Riemann wants to characterise the “general concept of magnitudes of several dimensions”, and the different “metrical relations” one can attach to these magnitudes. It is therefore solely a pure analysis, independent of any empirical consideration. More precisely, in part I., Riemann introduced what became the concept of differential manifold. This concept was developed in the following decades, notably with the emergence of set-topology and the major contributions of Poincaré in algebraic topology.[9]

[5] For other points of view on the same question –why did Weyl abandon Riemann-Finsler methods?–, see also Coleman and Korté (2001), Scholz (2004b, 2001).

[6] See Bernard (2013).

[7] Within the extensive literature on this point, we can first refer to Bonola (1912) and Riemann (1898).

[8] (Weyl, 1923a, pp. 8–9), Riemann (1919), etc.

[9] (Riemann, 1919, note 1), Scholz (1980).

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