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Essay review

The role of a posteriori mathematics in physics

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ABSTRACT

The calculus that co-evolved with classical mechanics relied on definitions of functions and differentials that accommodated physical intuitions. In the early nineteenth century mathematicians began the rigorous reformulation of calculus and eventually succeeded in putting almost all of mathematics on a set-theoretic foundation. Physicists traditionally ignore this rigorous mathematics. Physicists often rely on a posteriori math, a practice of using physical considerations to determine mathematical formulations. This is illustrated by examples from classical and quantum physics. A justification of such practice stems from a consideration of the role of phenomenological theories in classical physics and effective theories in contemporary physics. This relates to the larger question of how physical theories should be interpreted.

The a priori/a posteriori distinction, used by Kant, plays no role in mathematics. It does, however, play a role in philosophical interpretations of mathematically formulated theories. Philosophers of science have developed competing accounts of what scientific theories should be. In a syntactic account a scientific theory is presented, or reconstructed, as a deductive system based on axioms and employing both logical and mathematical rules of deduction. In a semantic account a theory is presented as a mathematical structure, such as phase space or Hilbert space, interpreted through a family of models. In both programs interpreting a theory is a matter of imposing a physical interpretation on a mathematical formalism. The math plays an a priori role. Both approaches insist that the mathematics used must have a validity independent of any physical interpretation imposed on it. The math, accordingly, must conform to rigorous standards.

The practice of many physicists often runs contrary to this interpretative methodology. This happens in two basic ways. The first is by beginning with physical assumptions and letting the physics determine the type of math used in the theory formulation. The second concerns justification, rather than selection. Physicists often justify mathematical arguments on physical rather than mathematical grounds. In both cases the math plays a methodologically a posteriori role. The criticism that such math is not rigorous is effectively countered by the claim: Too much rigor leads to rigor mortis. We will consider some examples of this practice in both classical and quantum physics and the reflect on their significance. We begin by considering the conceptual matrix from which calculus emerged.

1. The mathematics of classical physics

Pythagoras, Plato, and their disciples speculated on mathematical forms having some kind of existence independent of physical reality, or a pure a priori math. Aristotle's account of subalternation assigned mathematics a more a posteriori role in physical explanations. Arithmetic and geometry were regarded as idealizations derived from physical reality by abstraction and idealization: of numbers from units and of geometrical forms from physical shapes. Neither the Greeks, nor the Alexandrians, nor their Arabic successors ever developed a quantitative science of qualities. At the start of the Scientific Revolution, in the early Seventeenth century, a quantitative treatment of qualities was a common practice. A sketchy history can bring out the conceptual factors involved in the transition.[1]

In *de Interpretatione* (chap. 1) Aristotle developed the idea that things causally determine our concepts of them. The most basic concepts fit in his ordered list of categories: substance, quantity, quality, relation, place, time, situation, state, action, and passion. The first three categories have a conceptual ordering that determines the way quantities are treated. A quality, such as color

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[1] See MacKinnon 2012 for a more detailed history.

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presupposes extension, which in turn presupposes a substance that is extended and colored. A discussion of the quantity of a quality perverts the proper conceptual ordering. Aristotle's doctrine of categories was transmitted to medieval Scholastics, before the revival of Aristotelianism, through translations of Porphry's *Isagoge*, which summarized the doctrine.

Theological problems required a discussion of the quantity of qualities. The accepted doctrine, vividly illustrated in Dante's Divine Comedy that one's post-death assignment to a particular level in heaven, purgatory, or hell, depended on the degree (or lack) of sanctifying grace at the moment of death. Grace was regarded as a property of the soul, albeit a supernatural one. This property required quantification. Thomas Aquinas seems to have been the first to explicitly treat the quantification of qualities. (Summa Theologiae, 1,Q. 42, a. 1, ad 1) Under the later Nominalists this matured into a doctrine of the intensification and remission of qualities. This supplies the pivot transforming Aristotelian natural philosophy into mathematical physics. Newton's intermingling of physical and mathematical concepts presents a distinctive problem. As he explained it (Principia, p.38, Newton 1952a) the demonstrations are shorter by the method of indivisibles, his version of differential calculus, but since he deemed this harsh he followed the general method of ratios. Since his geometrical method now seems harsher we will rely on his theory of indivisibles. Quantities that vary continuously are called ’fluents', while their rates of change are called ’fluxions'. The basic problem of differentiation is: given the ratio of two fluents, to find the ratio of their fluxions.[2] This relied on physical concepts: “And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of their quantities not before they vanish, nor afterward, but with which they vanish.” (Principia, p. 39) This physical justification led both Newton and Euler to express the ratios with which quantities vanish as $0/0 = n$.

Euler tried to put calculus on an analytic, rather than a geometric basis. Here ’analytic' loosely means a function that can be expressed by a simple formula, $y = f(x)$, or by a Taylor expansion. Yet, he too relied on expressions of the form, $0/0 = n$ for the ratio of vanishing quantities. As Kline (Chap 13) has shown, 17th century scientists realized that the new calculus lacked an adequate foundation. Their reliance on calculus as a tool was effectively justified by a mixture of physical and theological considerations. Using calculus to solve physical problems led to correct results. Kepler and Galileo developed the idea that God created the world in accord with math-ematical forms. When Maupertuis introduced his Least Action Principle in 1744, he relied on a theological justification. The laws of matter must possess the perfection worthy of God's creation. Later physicists retained the a priori idea of a world fashioned in accord with mathematical forms even when they discounted the theological justification.

The co-evolution of physics and mathematics reached a branching point in the early nineteenth century. Two men led the differing developments. Laplace pioneered a new style of mathematical physics. In place of Lagrange's analytic mechanics Laplace developed a style that Poisson later dubbed ‘physical mechanics’. Even by the standards of his time his math was not rigorous. He used approximations and power series in which he regularly dropped terms that were deemed insignificant on physical grounds. He treated math as a tool, not a system. His younger contemporary, Cauchy, instituted a program of developing calculus with no explicit reliance on physical notions. Like Gauss and Fourier Cauchy continued to work on physics problems and this work inspired new developments in mathematics. However, mathematicians, following Cauchy's lead, abandoned physicalistic reasoning and geometric foundations in calculus in favor of analysis, and eventually set-theoretic foundations. (See Grattan-Guinness, 1980). These two trends epitomize what we have been referring to as a posteriori and a priori mathematics.

A simplified comparison of the divergent formulations of calculus can bring out the difference by focusing on three terms, ‘function’, ‘infinitesimal’ and ‘continuous’. In the old formulation[3] calculus is concerned with quantities that vary continuously and so can take on all possible values within boundary limits. This may be extended to quantities that have a limited number of discontinuities. A function expresses the relation between one quantity, the dependent variable, and another, the independent variable. It can usually be expressed through a simple analytic formula. $y = f(x)$, The basic unit of change for a continuous variable is an infinitesimal, or a differential. Infinitesimals are treated like quantities in the sense that $x + dx$ is a legitimate addition. Similarly, a derivative may be interpreted as the ratio of two differentials at the opposite extreme a set-theoretical formulation a function f: $S \longrightarrow T$, is a mapping that assigns to each element, s, of the domain, S, an element f(s)of the range, T. Abstract algebra is a broad topic. We will simply consider a set-theoretic formulation of operations. For this purpose, we can consider an algebra an ordered set, (A, o), with one or more operations. A function f: $(A, o) \longrightarrow (A', o')$ maps elements of A onto A′ and also carries the operation o into the operation o’. The functions of special concern here are homeomorphisms, in which the image of A is a proper subset of A‘. Finally, if mathematical continuity is not based on any notion of continuous quantities, then the fact that an interval is treated as a non-denumerable infinity of elements does not determine its length, differentiability, or integrability. Borel sets replace intuitive notions of continuity by beginning with point sets and then constructing aggregates that cover the interval. A Borel space is a set M with a σ −algebra. It elements are Borel sets. This is the smallest family of subsets of $\Re$ that includes the open sets and is closed under complementation and countable intersections. A measure is a function whose domain is some class of sets and whose range is an aggregate of non-negative real numbers. Borel sets supply a constructive method of assigning measures to sets so that the measure of an interval is the same as its length; congruent sets have equal measures; and the measure of a countable union of non-overlapping sets is equal to the sum of the measures of the individual sets. Borel sets do not presuppose physical continuity. Infinitesimals play no role in a set-theoretical formulation. The concept of an infinitesimal as a number greater than 0, but smaller than any assignable number, does not accord with the Archimedean axiom: For any numbers a, b, where a and b are positive numbers and $a < b$, there exists an n, a natural number, such that $na > b$.

Set theory provides a foundation for mathematics, one with well-known difficulties. Other foundations have been proposed, such as category theory, in which maps are basic elements. We will briefly consider two proposed foundations that allow infinitesimals. The first is the non-standard analysis developed primarily by Abraham Robinson, a severe critic of established set theory. (Robinson 1966) In non-standard theory set theory the natural numbers, N, are embedded in a larger set *N, which includes infinitely large numbers. This process is extended from the natural to the reals by embedding $\Re$ in $^*\Re$. The inverse of the infinitely large numbers in $^*\Re$ are infinitesimals. The second method of

[2] This is developed in Newton's "Tractatus de Quadruture Curvarum", first published in 1704. It is translated in Whiteside, pp.

[3] This is presented in old calculus books such as the highly popular texts by Granville, later Granville and Smith, and finally Granville, Smith, and Longley, published between 1929 and 1962.

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