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Adaptation of a one-step worst-case optimal univariate algorithm of bi-objective Lipschitz optimization to multidimensional problems

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ABSTRACT

A bi-objective optimization problem with Lipschitz objective functions is considered. An algorithm is developed adapting a univariate one-step optimal algorithm to multidimensional problems. The univariate algorithm considered is a worst-case optimal algorithm for Lipschitz functions. The multidimensional algorithm is based on the branch-and-bound approach and trisection of hyper-rectangles which cover the feasible region. The univariate algorithm is used to compute the Lipschitz bounds for the Pareto front. Some numerical examples are included.

1. Introduction

The problem of bi-objective non-convex optimization

$$\min_{x\in A} f(x), \quad f(x) = (f^1(x), f^2(x))^T, \quad A \subset \mathbb{R}^d, \tag{1}$$

is considered, where the properties of the objective functions and of the feasible region will be defined later. Theoretically the solution to problem (1) consists of two sets: $P(f,A)_O$, the Pareto optimal solutions in the space of objectives (in the bi-objective case $P(f,A)_O$ is frequently called the Pareto front), and $P(f,A)_D$, the set of the Pareto optimal decisions in A [7]. Analytically these sets can be found only in very specific cases. We are interested in a discrete representation of $P(f,A)_O$ for a class of non-convex objective functions.

The construction of an optimal algorithm in a broad class of global optimization algorithms is difficult [22]; nevertheless some semi-optimal solutions usually are possible. We assume that the feasible region is a hyper-rectangle $A = \{x : a_i \leqslant x_i \leqslant b_i,\ i = 1, \ldots, d\}$, and the objective functions are Lipschitz continuous. For the arguments in favor of the Lipschitz model we refer to [5,6,10,12,17]. For the alternative approaches in global optimization we refer, e.g. to [4,20,21]. In the present paper an algorithm is proposed hybridizing branch-and-bound approach, the concept of one-step worst-case optimality with respect to the class of Lipschitz functions, and trisection of hyper-rectangles covering the feasible region; the latter was originally proposed in [14,15]. For the review on branch and bound approach in global (including multi-objective) optimization we refer to [3,5,13,19]. The one-step optimality criterion prevails in the development of optimal algorithms of global optimization, see e.g. [2,8,11,18,19,23]. The concept of worst-case optimality, which is a standard in the theory of algorithms [1], is implemented e.g. in [25] to develop a univariate bi-objective optimization algorithm for Lipschitz objective functions.

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We start with an analysis of the possibility to generalize for the multidimensional case the results of [25] concerning the univariate ($d = 1$) bi-objective optimization. The problem of construction of the worst-case optimal algorithm for a discrete representation of Pareto fronts of problems with Lipschitz continuous objectives is considered in [24], where the optimal adaptive (sequential) algorithm is shown reducible to the passive optimal algorithm which in turn is coincident with covering of the feasible optimization region by the balls of the minimum radius. We refer to [25] for the arguments that from the applications point of view the concept of sequential one-step optimality seems appropriate. In the present paper a special class of the so called diagonal algorithms, which are in detail described in [16], is considered. The feasible region is subdivided by a diagonal algorithm into decreasing hyper-rectangles. A hyper-rectangle is selected for subdivision on the base of a criterion which depends on the objective function values at the endpoints of the diagonal. A subdivision rule is included into the description of the diagonal algorithm. Our goal is to define a criterion of the selection of a hyper-rectangle for subdivision.

2. Lipschitz bound for the Pareto frontier

Let $\Phi(L)$ be a class of vector-valued Lipschitz continuous functions where $L = (L^1, L^2)^T$ is the vector of Lipschitz constants with respect to the city-block metric, i.e. for $f(x) = (f^1(x), f^2(x))^T \in \Phi(L)$ the following inequalities are valid

$$|f^k(x) - f^k(t)| \leqslant L^k \cdot \|x - t\|, \quad k = 1, 2, \tag{2}$$

where $x \in A,\ t \in A,\ L^k > 0,\ k = 1, 2$, and $\|x - t\| = \sum_{i=1}^{d} |x_i - t_i|$.

The class of Lipschitz continuous functions is advantageous for constructing global minimization algorithms because of relatively simply computable lower bounds for the function values. The availability of such bounds enables a theoretical assessment of the quality of a discrete representation of the Pareto front for bi-objective Lipschitz optimization.

Let $a(r) \in A$ and $b(r) \in A$ be the end points of a diagonal of a hyper-rectangle A_r; without loss of generality it is assumed that $a_i(r) < b_i(r),\ i = 1, \ldots, d$. As follows from (2) the functions $g^k(x, A_r),\ k = 1, 2$, define the lower bounds for $f^k(x),\ x \in A_r$:

$$g^k(x, A_r) = \max\left(f^k(a(r)) - L^k \sum_{i=1}^{d} (x_i - a_i(r)), f^k(b(r)) - L^k \sum_{i=1}^{d} (b_i(r) - x_i)\right). \tag{3}$$

The Pareto front of the bi-objective problem

$$\min_{x \in A_r} g(x, A_r), \quad g(x, A_r) = \left(g^1(x, A_r), g^2(x, A_r)\right)^T, \tag{4}$$

is denoted by $V_r = V(f(a(r)), f(b(r)), A_r)$.

Lemma 1. *No element of V_r is dominated by a vector $f(x),\ x \in A_r$.*

Proof. Let us assume contrary, that there exist $x \in A_r$ and $y \in V_r$ such that $z = f(x) \succ y$. Since (3) implies that $g(x, A_r) \succeq z$, and V_r is the subset of non-dominated elements of the set of $\{f(x) : x \in A_r\}$, there exists an element $v \in V_r$ such that

$$v \succeq g(x, A_r) \succeq z \succ y. \tag{5}$$

However the obtained relation of dominance of v over y can not be truth since both, v and y, are elements of the Pareto front V_r. Therefore the assumption made at the beginning of the proof is not truthful, and the proof is completed. □

The following definition is a natural sequel of Lemma 1:

Definition 1. V_r is called a local lower Lipschitz bound for $P(f, A)_O$.

Definition 2. The Pareto front of the set $\bigcup_{r=1}^{R} V_r$ is called a lower Lipschitz bound for $P(f, A)_O$ and is denoted by $\mathbf{V}(\mathbf{Y_R}, \mathbf{A_{[R]}})$, where $\mathbf{Y_R} = \{f(a(r)), f(b(r)) :\ r = 1, \ldots, R\}$, $\mathbf{A_{[R]}}$ denotes the set of hyper-rectangles which constitute partition of A, and $a(r)$ and $b(r)$ are the end points of the diagonals of the mentioned hyper-rectangles.

Let us consider the bi-objective minimization problem on a line segment $\tilde{A}(r)$

$$\min_{x \in \tilde{A}(r)} g(x, A_r), \tag{6}$$

where $\tilde{A}(r)$ denotes the diagonal of A_r. The Pareto front of (6) is denoted by $\tilde{V}(r)$.

We show illustration of two objective functions f^1 and f^2 in Fig. 1. The functions are multimodal as can be seen from surface and contour plots. We also show city-block bounding functions computed taking into account function values at the endpoints of the diagonal of the square $[0, 1]^2$ and different slopes for each objection function. If one views from a certain angle these lower bounding functions are seen as line segments (see lower plots). These line segments can be interpreted as bounding functions over diagonal.

Taking such bounding functions into account, the graphical representation of $\{g(x, A_r) : x \in A_r\}$ can be made similarly to [24]. We show such a graphical representation in Fig. 2, where $f(a(r)) = y = (y_1, y_2)^T,\ f(b(r)) = z = (z_1, z_2)^T$, and $\tilde{V}(r)$ is