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An algorithm for computing a neighborhood included in the attraction domain of an asymptotically stable point

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ABSTRACT

Many methods exist to detect stable equilibrium points x^* of nonlinear dynamical systems $\dot{x} = f(x)$. Most of them also prove the existence of a neighborhood $\mathcal{N}$ of x^* such that all trajectories initialized in $\mathcal{N}$ converge to x^*. This paper provides a numerical method combining Lyapunov theory with interval analysis which makes to find a set $\mathcal{N}$ which is included in the attraction domain of x^*.

1. Introduction

Consider a nonlinear dynamical system described by a differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field. The point x^* is an equilibrium point if $f(x^*) = 0$. To find the equilibrium points it suffices to solve n nonlinear equations with n unknowns. This can be solved using elimination theory-based methods [18], or any local numerical algorithm [20]. A point x^* is asymptotically stable if for all neighborhood $\mathcal{M}$ of x^*, there exists a neighborhood $\mathcal{N}$ of x^* such that all trajectories initialized in $\mathcal{N}$ converge to x^* and remain inside $\mathcal{M}$.

From the theoretical point of view, the Hartman–Grobman theorem states that if f is sufficiently regular around a hyperbolic equilibrium state x^* then there exists a local homeomorphism between the solutions of the $\dot{x} = f(x)$ and its linearization $\dot{x} = Df(x^*)(x - x^*)$. In other words, the qualitative behavior of the dynamical system f around x^* is the same that of $Df(x^*)$. Therefore, the existence of $\mathcal{N}$ is usually provided by studying the eigenvalues of the Jacobian matrix of f at x^*. Interval based methods have already been used to study the stability of dynamical systems. In the case of linear system, a classical result from control theory states that the origin (which is always an equilibrium state) is stable if and only if all roots of the characteristic polynomial of f have a negative real part. Such a polynomial is said to be Hurwitz stable. In [16], Khraritonov gives a necessary and sufficient effective condition to the Hurwitz stability of a polynomial with interval coefficients. When f is linear with unknown bounded coefficients (i.e. f can be represented by a matrix whose entries are intervals), the Khraritonov's condition only offers a sufficient condition to check that the origin is stable. More recently, Wang and al [17] determine a necessary and sufficient effective condition to the Hurwitz stability of an interval matrix.

The present paper deals with nonlinear dynamical system. Contrary to the linear case, the stability of an equilibrium state is, most of the time, only local: the trajectories must be initialized sufficiently close to the equilibrium state x^* to converge to x^*. The set of initial states for which the trajectory converges to x^* is the *attraction domain* of x^*. The main contribution of this paper is an algorithm which provides a neighborhood $\mathcal{N}$ of x^* included in the attraction domain of x^*.

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Given an equilibrium for a dynamical system, we have the well-known connection with the linearization near the stationary point. By studying this linearization it is more or less straightforward to construct such neighborhoods $\mathcal{N}$, see for example [3,4]. The approach to be considered, based on Lyapunov theory and interval analysis, also proves existence and uniqueness of an asymptotically stable equilibrium state x^* even if we only have a rigorous enclosure of x^*.

The paper is organized as follows. Interval analysis is briefly presented in Section 2. Section 3 provides a method and a sufficient condition to check that a real valued function is positive. In Section 4, we combine interval analysis and Lyapunov analysis in an algorithm that is able to solve our stability problem. Finally, an example illustrates our approach in Section 5.

2. Interval arithmetic

This section introduces notations and definitions related to interval analysis. An interval $[\underline{x}, \overline{x}]$ of $\mathbb{R}^n$ is a set which can be written as $\{x \in \mathbb{R}^n,\ \underline{x} \leqslant x \leqslant \overline{x}\}$ with $\underline{x}$ and $\bar{x}$ in $\mathbb{R}^n$. Here the relation $\leqslant$ has to be understood component-wise. Note that this definition implies that intervals are bounded. The set intervals is usually denoted by $\mathbb{IR}^n$.

Definition 1. A map $[f] : \mathbb{IR}^n \to \mathbb{IR}^m$ is said to be an inclusion map of $f : \mathbb{R}^n \to \mathbb{R}^m$ if $\forall [x] \in \mathbb{IR}^n, f([x]) \subset [f]([x])$ (where $f([x]) = \{f(x) | x \in [x]\}$).

Interval arithmetic [1,13,19] provides an effective method to build inclusion maps. In [5], Neumaier proves that it is always possible to find an inclusion map $[f]$ when f is defined by an arithmetical expression. This possibility to enclose the image of an interval $[x]$ under f is powerful. Indeed, let us suppose that $0 \notin [f]([x])$, one can conclude that $\forall x \in [x],\ f(x) \neq 0$. On the other hand, if $0 \in [f]([x])$, this does not imply that $\exists x \in [x] | f(x) = 0$. Fig. 1

Since Moores works [1,2] that introduced interval arithmetic, many algorithms have been developed in different areas, for example in global optimization [7], non-linear dynamical systems, etc. As interval analysis provides rigorous methods, these algorithms can prove mathematical assertion. For instance, in 2003, Hales launched the "Flyspeck project" ("Formal Proof of Kepler") in an attempt to use computers to automatically verify every step of the proof (partially based on interval analysis) of the Kepler's conjecture. Another important example is a generalization of the Newton method called Interval Newton method. This method can be applied to find all zeros of a given differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$. The interval Newton method creates a sequence of intervals containing zeros of f and has very interesting properties: combined with Brouwer fixed point theorem, it can prove existence and uniqueness of a zero of f [6,14].

Note that the set of inclusion maps of a given $f : \mathbb{R}^m \to \mathbb{R}^n$ can be partially ordered by the relation: $[f]_1 \leqslant' [f]_2 \iff \forall [x] \in \mathbb{IR}^n,\ [f]_1([x]) \subset [f]_2([x])$. Due to the fact that the available inclusion map is rarely minimal (related to $\leqslant'$), interval analysis cannot basically be used to prove the assertion $\forall x \in [x],\ f(x) \geqslant 0$ in the case of existence of $x_0 \in [x]$ such that $f(x_0) = 0$. The next section shows how such a proof can be done by combining interval computation with algebra calculus.

3. Sufficient condition to check $f \geqslant 0$.

This section proposes a theorem which provides a sufficient condition to check the following assertion for a given differentiable real valued function f: $\forall x \in [x],\ f(x) \geqslant 0$. The main idea is close to the second derivative criterion classically used in optimization. Then, an algorithm based on the proposed theorem and interval analysis is presented. Let us recall that a symmetric real matrix A is positive definite if $\forall x \in \mathbb{R}^n - \{0\},\ x^T A x > 0$. In this paper, the set of positive definite symmetric $n \times n$ matrices is denoted by S^{n+}.

Theorem 1. *Let* $f \in C^\infty([x] \subset \mathbb{R}^n,\ \mathbb{R})$. *If there exists* $x^* \in [x]$ *such that* $f(x^*) = 0$ *and* $Df(x^*) = 0$, *and* $\forall x \in [x],\ D^2 f(x) \in S^{n+}$, *then* $\forall x \in [x],\ f(x) \geqslant 0$ *and* $f(x) = 0 \Rightarrow x = x^*$.

Proof. The assertion $\forall x \in [x],\ D^2 f(x) \in S^{n+}$ implies that f is a strictly convex function defined on a convex set $[x]$. Since $Df(x^*) = 0$, one can conclude that $\inf_{x \in [x]} f(x) \geqslant f(x^*) = 0$. The proof of uniqueness is by reduction to a contradiction. Suppose that there exists $x^{**} \in [x]$ such that $f(x^{**}) = 0$ and $x^{**} \neq x^*$. As f is strictly convex, one has

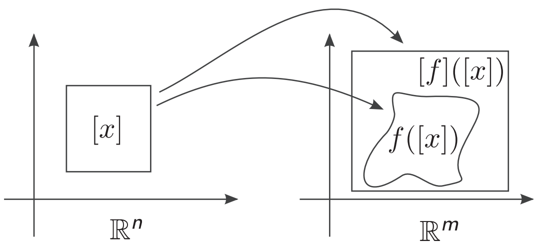

Fig. 1. Illustration of inclusion function.