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Reprint of Solution of Ambrosio–Tortorelli model for image segmentation by generalized relaxation method ☆,☆☆

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ABSTRACT

Image segmentation addresses the problem to partition a given image into its constituent objects and then to identify the boundaries of the objects. This problem can be formulated in terms of a variational model aimed to find optimal approximations of a bounded function by piecewise-smooth functions, minimizing a given functional. The corresponding Euler–Lagrange equations are a set of two coupled elliptic partial differential equations with varying coefficients. Numerical solution of the above system often relies on alternating minimization techniques involving descent methods coupled with explicit or semi-implicit finite-difference discretization schemes, which are slowly convergent and poorly scalable with respect to image size. In this work we focus on generalized relaxation methods also coupled with multigrid linear solvers, when a finite-difference discretization is applied to the Euler–Lagrange equations of Ambrosio–Tortorelli model. We show that non-linear Gauss–Seidel, accelerated by inner linear iterations, is an effective method for large-scale image analysis as those arising from high-throughput screening platforms for stem cells targeted differentiation, where one of the main goal is segmentation of thousand of images to analyze cell colonies morphology.

1. Introduction

Image segmentation is one of the central tasks in image processing and has the aim of partitioning an image into its constituent regions or objects, leading to also identify the boundaries of these objects. Usually, two types of techniques are used for image segmentation: the *discontinuity-based approach*, where main aim is to identify the set of pixels in which intensity function has high gradients or jumps (*the edges*) or the *similarity-based approach*, where the aim is to identify regions with similar characteristics in intensity function. The problem can be formulated in terms of a variational model, i.e., in terms of an energy minimization criterion, which in some sense merges features of both the above approaches. The original variational model for segmentation is a *free-discontinuity problem* formulated by Mumford and Shah [17] that proposed to look for a piecewise smooth approximation u of the original image function f, with u discontinuous across a closed set K included

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in the image domain Ω. In more details, let $\Omega \subset \Re^2$ be a bounded open set and $f \in L^\infty(\Omega)$ the observed gray-level image, the problem consists in the minimization of the following functional:

$$E(u,K) = \int_\Omega (u-f)^2 dxdy + \beta \int_{\Omega \setminus K} |\nabla u|^2 dxdy + \alpha |K|, \tag{1}$$

where $u \in C^1(\Omega \setminus K), K \subset \Omega$ is a closed set whose $|K|$ is the length of K, and α and β are positive coefficients.

This problem is not simple to deal with due to the presence of the last term. However, an extensive theory has been developed since its introduction and many efforts have been devoted to approximate the problem with a relaxed one which can be solved by classical approach of Calculus of Variations. On the other hand, since the functional is not convex, large efforts have been also devoted to relax the model in order to get convexity. It is beyond the focus of this work to discuss theoretical aspects or advantages and drawbacks of the Mumford–Shah (MS) model, however, we can observe that it is the most general way to formulate a segmentation problem and the well-based theory developed during the last 20 years motivates its use to obtain robust and accurate software modules for large-scale analysis. For classical books and recent reviews on theory, numerical approximations and applications of the MS model we refer to [1,2,6,16,22].

One of the most general approximations of the MS model was proposed by Ambrosio and Tortorelli who showed that the model can be approximated, in the sense of Γ-convergence, by a sequence of elliptic functionals [3,4]. For details on Γ-convergence and its role in the study of asymptotic variational problems we refer to [9]. Ambrosio and Tortorelli introduced a new variable $0 \leqslant z \leqslant 1$, which controls $|\nabla u|$ and gives an approximate representation of the set K in a tubular neighbourhood of radius ϵ of the minimizer K. The possibility to approximate the measure of K by an elliptic functional depending on z leads to the following sequence of functionals depending on ϵ:

$$E_\epsilon(u,z) = \int_\Omega (u-f)^2 dxdy + \beta \int_\Omega z^2 |\nabla u|^2 dxdy + \alpha \int_\Omega \left(\epsilon |\nabla z|^2 + \frac{(z-1)^2}{4\epsilon} \right) dxdy, \tag{2}$$

which Γ-converges to $E(u,K)$ for $\epsilon \to 0$. Note that $E_\epsilon(u,z)$ remains not convex and a drawback of this approximation is its dependence on the choice of a good ϵ parameter in numerical solution. In [3,21] the reader can find a clear discussion, which we do not report here for sake of brevity, on the behaviour of the function z introduced by Ambrosio and Tortorelli in their work. However, we notice that this function is forced to tend to 1 as ϵ tends to 0 because the term $(z-1)^2$ is positive and vanishes only for $z = 1$; furthermore, the term z^2 must go to zero near the discontinuities of u, where $|\nabla u|$ is large, to keep bounded the second term in (2). Therefore, z makes a sharp transition to zero inside a neighbourhood of the set K of thickness of radius ϵ and an accurate numerical solution should require a suitable spatial discretization of the model to well resolve the above tubular region.

Some efforts have been devoted to minimize the functional in (2) by finite-element approximations, see for example [7,8]. On the other hand, the form of the functional in (2) allows us to apply the classical approach of Calculus of Variations, i.e. writing Euler–Lagrange equations to obtain stationary solutions. Numerical solutions of the Euler–Lagrange equations associated to (2) are usually based on finite-difference discretizations of the equations and alternating minimization schemes are obtained by applying gradient descent iterations [6,19,22]. In [21] the authors proposed to use the non-linear Gauss–Seidel method for solving discrete equations arising from a finite-difference approximation of (2). In this paper we show that non-linear Gauss–Seidel relaxation, coupled with a fixed-point approach producing inner linear systems at each application of a basic step of the non-linear method to a discrete form of the equations, is very effective for large-scale image segmentation. Inner iterations largely reduce the number of non-linear iterations already for very low accuracy requests on inner solutions but also improves robustness when those accuracy requests are increased, leading to reliable, flexible and efficient solver. We discuss convergence results and computational cost of the method in the analysis of 2D gray-scale images of embryonic stem cells colonies, since our final aim is to develop an efficient segmentation module for automatic screening of colonies morphology useful to discriminate different phenotypic transitions.

The paper is organized as follows. In Section 2 we introduce a finite-difference discretization of the Euler–Lagrange equations for model (2). In Section 3 we briefly describe the non-linear Gauss–Seidel method when applied to a discrete form of (2), then we describe our fixed-point approach coupled with the non-linear relaxation and outline the linear multigrid solver which can be used to accelerate inner convergence. In Section 4 we present results obtained on real images of cultures of mouse embryonic stem cells. An extensive discussion on the performance of our method, in terms of robustness, accuracy, computational costs and comparison with standard gradient descent methods, varying model parameters, is reported. Concluding remarks and future work are included in Section 5.

2. Finite-difference discretization of Euler–Lagrange equations

Euler–Lagrange equations for Ambrosio–Tortorelli model are the following system of two coupled elliptic PDEs, associated with Neumann boundary conditions:

$$\begin{cases} 2(u-f) - 2\beta \nabla \cdot (z^2 \nabla u) = 0 \\ 2\beta z |\nabla u|^2 - 2\alpha\epsilon \nabla^2 z + \frac{\alpha}{2\epsilon}(z-1) = 0 \end{cases} \quad (x,y) \in \Omega, \tag{3}$$