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On the existence of conservation law multiplier for partial differential equations

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ABSTRACT

In this paper, we use the property of nonlinear self-adjointness with differential substitution to study the existence of conservation law multiplier for partial differential equations (PDEs). Firstly, we give a sufficient and necessary condition for the existence of the multipliers involving only independent and dependent variables, which is the nonlinear self-adjointness of the studying PDEs. Secondly, a necessary condition for the existence of the multipliers involving derivatives is given for the general evolution PDEs, which is the nonlinear self-adjointness with differential substitution. Finally, applications of multiplier and nonlinear self-adjointness with differential substitution methods to nonlinear telegraph equations and a class of Korteweg-de Vries (KdV) type equations are performed and different types of conservation laws are constructed.

1. Introduction

The classical Noether's theorem provides an elegant and constructive way to obtain conservation laws of PDEs which admit a variational principle. A system of PDEs has a variational principle if and only if its linearized system is self-adjoint [1]. However, a limitation of Noether's theorem is that it restricts to variational systems, or says it depends on the existence of Lagrangian, where the majority of the PDEs arising in applications do not hold this property such as scalar odd order evolution PDEs. Thus the methods which do not count on the existence of Lagrangian are developed to construct conservation laws of the PDEs which do not admit a variational form (See [1–8] and references therein).

One of the elementary methods is the direct method developed by Laplace, which obtains conservation laws by means of the structure of PDEs directly [2]. Steudel writes a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations [3]. Thus in order to determine a conservation law with this approach one has to find the related multiplier. In [1], it was shown how to find the multipliers of conservation law for a given system of PDEs via the variational derivative. In [5,6], the direct construction method was presented to obtain the multipliers and, through an integral formula, the corresponding local conservation laws for a given system of PDEs. Within this method, the multipliers are found from the determining equations followed from Euler operators, and then are used to derive explicit expressions of the corresponding conservation laws. In [7], Kara and Mahomed imposed a natural symmetry condition which together with the direct method simplifies the solution procedure for the determination of a conservation law. They also presented a partial Noether approach for the differential equations with or without a Lagrangian [8]. Comparisons of these methods for some PDEs were performed in [9,10].

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Recently, the concept of nonlinear self-adjointness[11,12], including strict self- adjointness [13], quasi self-adjointness [14] and weak self-adjointness [15] stated earlier, provided a feasible method to construct conservation laws of PDEs without having a variational structure [16–18]. The main idea of the method, which traced back to the literatures [19,20] and references therein, is to turn the system of PDEs into Lagrangian equations by artificially adding additional variables, then to apply the theorem proved in [21] to construct local and nonlocal conservation laws. Approximate nonlinear self-adjointness and approximate conservation laws of perturbed PDEs were considered in [11,22] and its relation with nonlinear self-adjointness was studied in [23].

The purpose of the paper concentrates on the existence of conservation law multiplier by means of the property of nonlinear self-adjointness with differential substitution. We find that the key of both two methods is to solve the corresponding determining systems which are obtained by means of Euler operator. Thus inspired by the facts, we show the following two results:

1. The determining system for the multipliers involving only independent and dependent variables is identical to the one for the substitution of nonlinear self-adjointness.
   Consequently, it demonstrates that nonlinear self-adjointness of PDEs is equivalent to the existence of the multipliers involving only independent and dependent variables. Then after solving the same determining system, different types of conservation laws are constructed by two different formulae.
2. For the system of general evolution PDEs, a necessary condition for the existence of multipliers involving derivatives is given by means of nonlinear self-adjointness with differential substitution. We show that the set of differential substitutions of nonlinear self-adjointness is the same as the one of adjoint symmetries, which contains multipliers as a subset.

The remainder of the paper is arranged as follows. In Section 2, some related basic notions and principles are reviewed and the main results are given. In Section 3, nonlinear telegraph equations and a class of KdV type equations are considered to illustrate the results. The last section contains a conclusion.

2. Main results

In this section, we first briefly recall some notions and principles related with multiplier method and nonlinear self-adjointness with differential substitution, and then give the main results of the paper.

2.1. Preliminaries

2.1.1. Notations

Let $x = (x^1, \ldots, x^n)$ be an independent variable set, $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m)$ be two dependent variable sets, $\varphi(x, u) = (\varphi^1, \ldots, \varphi^m)$ be a m-dimensional vector function, whose different order derivatives are denoted as follows

$$\Delta_{(1)} = \{\Delta^\sigma_{i_1}\},\ \Delta_{(2)} = \{\Delta^\sigma_{i_1 i_2}\},\ \ldots,\ \Delta_{(r)} = \{\Delta^\sigma_{i_1\ldots i_r}\},$$

where $\Delta^\sigma_{i_1\ldots i_s} = D_{i_1} D_{i_2} \ldots D_{i_s}(\Delta^\sigma)$ and $\sigma = 1, 2, \ldots, m$. The symbol "Δ" denotes the dependent variables u, v and vector function $\varphi(x, u)$. Hereinafter, D_i denotes the total derivative operator with respect to x^i. For example, for one dependent variable $u = u(x, t)$ with $x = x^1,\ t = x^2$, one has

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \cdots$$

Note that we will use these symbols and the summation convention for repeated indices throughout the paper if no special explanations are added.

Next, we take the following system of m PDEs with rth-order

$$E^\alpha(x, u, u_{(1)}, \cdots, u_{(r)}) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{1}$$

to recall some related notions and principles about multiplier method and nonlinear self-adjointness with differential substitution.

Definition 2.1 (*Conservation law [4]*). A conservation law of PDEs system (1) is a divergence form

$$D_i(C^i) = D_1(C^1) + \cdots + D_n(C^n) = 0$$

for all solutions of system (1), where $C^i = C^i(x, u, u_{(1)}, \ldots, u_{(q)})$ are called the fluxes of conservation law and the highest order derivative q presented in the fluxes C^i is called the order of a conservation law.