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Differential invariants for third-order evolution equations

C. Tsaousi [a], R. Tracinà [b], C. Sophocleous [a,*]

[a] Department of Mathematics and Statistics, University of Cyprus, Nicosia CY 1678, Cyprus
[b] Dipartimento di Matematica e Informatica, Viale A. Doria 6, 95125 Catania, Italy

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ABSTRACT

We consider a general class of third order evolution equations. We construct differential invariants with the employment of the infinitesimal method that using equivalence groups. We use the differential invariants to classify those equations from the class that can be mapped into a specific linear equation.

1. Introduction

Special cases of the general class of evolution equations

$$u_t = f(x, u, u_x, u_{xx})u_{xxx} + g(x, u, u_x, u_{xx}), \tag{1}$$

have been used to model successfully physical phenomena in Mathematical Physics. Such examples is the KdV equation

$$u_t = u_{xxx} + uu_x,$$

the modified KdV equation

$$u_t = u_{xxx} + u^2 u_x,$$

the $K(m,n)$ equations, which is a generalization of the above two equations,

$$u_t + \epsilon(u^m)_x + \frac{1}{n}(u^n)_{xxx} = 0$$

and the Harry-Dym equation

$$u_t = u^3 u_{xxx}.$$

A complete point symmetry classification of all third-order evolution equations of the form (1) which admit semi-simple symmetry algebras and extensions of these semi-simple Lie algebras by solvable Lie algebras is presented in [4].

The present paper is in the spirit of the work in Ref. [15], where differential invariants were constructed for the class of second-order evolution equations

* Corresponding author. Tel.: +357 22892630.
E-mail addresses: tsaousi.christina@ucy.ac.cy (C. Tsaousi), tracina@dmi.unict.it (R. Tracinà), christod@ucy.ac.cy (C. Sophocleous).

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$$u_t = f(x, u, u_x)u_{xx} + g(x, u, u_x)$$

and these invariants were used to construct those forms of the above class that can be mapped into the linear heat equation $u_t = u_{xx}$. In particular, here we consider the problem of finding differential invariants for the general class (1). Furthermore, we construct differential invariants for the two special cases of (1):

$$u_t = f(x, u, u_x)u_{xxx} + g(x, u, u_x, u_{xx}) \tag{2}$$

and

$$u_t = u_{xxx} + g(x, u, u_x, u_{xx}). \tag{3}$$

Galilei symmetry group classification for the class (3) was carried out recently in [7].

The theory of differential invariants of the Lie groups of continuous transformations play important role in mathematical modeling, non-linear science and differential geometry. Lie was the first to show [21] that every invariant system of differential equations [22], and every variational problem [23], could be directly expressed in terms of differential invariants. Furthermore Lie demonstrated [22], how differential invariants can be used to integrate ordinary differential equations, and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Lie's preliminary results on invariant differentiations and existence of finite bases of differential invariants were generalized by Tresse [33] and Ovsiannikov [27]. The general theory of differential invariants of Lie groups including algorithms of construction of differential invariants can be found, for example, in [25,27].

A simple method that introduced by Ibragimov [8] is employed to derive the desired differential invariants. Since then, there exists a continuing interest in this area of Mathematics and on this specific method. In fact, the method was adopted by several authors who used it to construct differential invariants for ordinary differential equations [1–3,10,14], for linear [9,11,16–19,24,35,37] or nonlinear [12,15,29–32,34] or systems of partial differential equations [36]. For example, Ibragimov [11] derived a solution of the Laplace problem that consists of finding all invariants of the linear hyperbolic equations

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0.$$

Namely, in addition to the two known invariants derived by Ovsiannikov [26], he constructed three new invariants. An alternative approach for deriving differential invariants is Cartan's method [6,25].

Equivalence transformations play an important part in the theory of invariants. Derivation of equivalence transformations for the class of equations under consideration is the first step towards to the target which is the determination of differential invariants. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods for calculation of equivalence transformations, the direct which was used first by Lie [22] and the Lie infinitesimal method which was introduced by Ovsyannikov [27]. Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group and also unfolds all form-preserving (admissible) transformations admitted by this class of equations. For recent applications of the direct method one can refer, for example, to references [38–41]. More detailed description and examples of both methods can be found in [13]. The method that we employ here to determine differential invariants requires the equivalence transformations to be in the infinitesimal form. Hence, we use the infinitesimal method to derive the desired equivalence transformations.

In what follows, we state certain definitions that we use in the paper. We call an equivalence transformation of a class of pdes, $E(x, t, u) = 0$, an invertible mapping of the variables t, x and u of the form

$$t' = Q(t, x, u), \quad x' = P(t, x, u), \quad u' = R(t, x, u) \tag{4}$$

that maps every equation of the class into an equation of the same class, $E(x', t', u') = 0$. For example, in the case of the class (1), an equivalence transformation maps (1) into

$$u'_{t'} = f'(x', u', u'_{x'}, u'_{x'x'})u'_{x'x'x'} + g'(x', u', u'_{x'}, u'_{x'x'}), \tag{5}$$

where the transformed functions f' and g' can, in general, be different from the original functions f and g. The set of all equivalence transformations forms the equivalence group.

A function of the form

$$J(t, x, u, u_x, u_{xx}, f, g, f_x, f_u, f_{u_x}, f_{u_{xx}}, g_x, g_u, g_{u_x}, g_{u_{xx}}, \ldots)$$

which remains invariant under the equivalence group is called differential invariant of order s of Eq. (1), where s denotes the maximal order derivative of f and/or g. If $J = J(t, x, u, u_x, u_{xx}, f, g)$, then is called invariant of order zero. An equation of the form

$$I(t, x, u, u_x, u_{xx}, f, g, f_x, f_u, f_{u_x}, f_{u_{xx}}, g_x, g_u, g_{u_x}, g_{u_{xx}}, \ldots) = 0$$

is called an invariant equation if it is invariant under the equivalence transformation modulus the equation.

In order to derive the continuous group of equivalence transformations of a class of pdes by means of the Lie infinitesimal invariance criterion [27], we search for the equivalent operator of the following form: