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Vector calculus in non-integer dimensional space and its applications to fractal media

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

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ABSTRACT

We suggest a generalization of vector calculus for the case of non-integer dimensional space. The first and second orders operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space are defined. For simplification we consider scalar and vector fields that are independent of angles. We formulate a generalization of vector calculus for rotationally covariant scalar and vector functions. This generalization allows us to describe fractal media and materials in the framework of continuum models with non-integer dimensional space. As examples of application of the suggested calculus, we consider elasticity of fractal materials (fractal hollow ball and fractal cylindrical pipe with pressure inside and outside), steady distribution of heat in fractal media, electric field of fractal charged cylinder. We solve the correspondent equations for non-integer dimensional space models.

1. Introduction

In general we can assume that space and space–time dimensions are D, which need not be an integer. Non-integer dimensional spaces and method of dimensional continuation are initially emerged in statistical mechanics and quantum field theory. Non-integer dimension $D = 4 - \varepsilon$ of space–time and ε-expansion are actively used in the theory of critical phenomena and phase transitions in statistical physics (for example, see [1,2]). Integration over non-integer dimensional spaces is used in the dimensional regularization method as a powerful tool to obtain exact results without ultraviolet divergences in quantum field theory [3–5]. In quantum theory, the divergences are parameterized as quantities with coefficients $\varepsilon^{-1} = (4 - D)^{-1}$, and then these divergences can be removed by renormalization to obtain finite physical values.

The axioms for integrals in non-integer dimensional space are suggested by Wilson in [6]. These properties are natural and necessary in applications in different areas [5]. Theory of integration in non-integer dimensional spaces has been suggested in [7,5,8]. Stillinger introduces [7] a mathematical basis of integration on spaces with non-integer dimensions. In [7] has been suggested a generalization of the Laplace operator for non-integer dimensional spaces also. In the book by Collins [5] the integration in non-integer dimensional spaces is formulated for rotationally covariant functions. The product measure method, which is suggested in [9], and the Stillinger's approach [7] are extended by Palmer and Stavrinou [8] to multiple variables and different degrees of confinement in orthogonal directions. In the paper [8] extensions of integration and scalar Laplace operator for non-integer dimensional spaces are suggested.

* Tel.: +7 4959395989.
E-mail address: tarasov@theory.sinp.msu.ru

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The scalar Laplace operators, which are suggested in [7,8] for non-integer dimensional spaces, have a wide application in physics and mechanics. Non-integer dimensional space has successfully been used as an effective physical description. The Stillinger's form of Laplacian first applied by He [10–13], where the Schrödinger equation in non-integer dimensional space is used and the real confining structure is replaced by an effective space, such that the measure of the anisotropy or confinement is given by the non-integer dimension. Non-integer dimensions is used by Thilagam to describe stark shifts of excitonic complexes in quantum wells [14], exciton–phonon interaction in fractional dimensional space [15], and blocking effects in quantum wells [16]. The non-integer dimensional space approach is used by Matos-Abiague [17–23] to describe momentum operators for quantum systems and Bose-like oscillator in non-integer dimensional space, the polaron effect in quantum wells. Quantum mechanical models with non-integer (fractional) dimensional space has been suggested by Palmer and Stavrinou [8], Lohe and Thilagam [24]. The non-integer dimensional space approach is used to describe algebraic properties of Weyl-ordered polynomials for the momentum and position operators [25,26] and the correspondent coherent states [27]. The Stillinger's form of Laplacian has been applied to the Schrödinger equation in non-integer dimensional space by Eid, Muslih, Baleanu, Rabei in [28,29], Muslih and Agrawal [30,31], by Calcagni, Nardelli, Scalisi in [32]. The fractional Schrödinger equation with non-integer dimensions is considered by Martins, Ribeiro, Evangelista, Silva, Lenzi in [33] and by Sandev, Petreska, Lenzi [34]. Recent progress in non-integer dimensional space approach includes the description of the scalar field on non-integer dimensional spaces by Trinchero [35], the fractional diffusion equation in non-integer dimensional space and its solutions are suggested in [36]. The gravity in fractional dimensional space is described by Sadallah, Muslih, Baleanu in [37,38], and by Calcagni in [39–41]. The electromagnetic fields in non-integer dimensional space are considered in [42–49].

Unfortunately, the basic articles [7,8] proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian are not considered in [7,8]. In the book [49] (see also [45–48]), the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Consideration only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional space for fractal media and material. For example, we cannot use the Stillinger's form of Laplacian for displacement vector field $\mathbf{u}(\mathbf{r},t)$ in elasticity and thermoelasticity theories. We cannot consider equations for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic fields $\mathbf{B}(\mathbf{r},t)$ for electromagnetic theory of fractal media by using continuum models with non-integer dimensional space.

In this paper, we propose a vector calculus for non-integer dimensional space and we define the first and second orders differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space. For simplification we consider rotationally covariant scalar and vector functions that are independent of angles. In order to derive the vector differential operators in non-integer dimensional space we use the method of analytic continuation in dimension. For this aim we get equations for these differential operators for rotationally covariant functions in $\mathbb{R}^n$ for arbitrary integer n to highlight the explicit relations with dimension n. Then the vector differential operators for non-integer dimension D are defined by analytic continuation from integer dimensions n to non-integer D. These proposed equations allows us to reduce D-dimensional vector differentiations to usual derivatives with respect to one variable $r = |\mathbf{r}|$. It allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to r. The proposed operators allows us to describe fractal materials and media in the framework of continuum models with non-integer dimensional spaces. In order to give examples of the possible applications, we consider continuum models of fractal media and materials in the elasticity theory in the heat theory, and in the theory of electric fields. The correspondent equations for non-integer dimensional space are solved.

2. Fractal media

The cornerstone of fractal media is the non-integer dimension [50] such as mass or charge dimensions [51,52]. In general, fractal media and materials can be treated with three different approaches: (1) Using the methods of "Analysis on fractals" [53–58] it is possible to describe fractal materials; (2) To describe fractal media we can apply fractional–integral continuous models suggested in [59–62,52] (see also [63–76]). In this case we use integrations of non-integer orders and two different notions such as density of states and distribution function [52]; (3) Fractal materials can be described by using the theory of integration and differentiation for a non-integer dimensional space [5,7,8].

The first approach, which is based on the use of analysis on fractal sets, is the most stringent possible method to describe idealized fractal media. Unfortunately, it has two lacks. Firstly, a possibility of application of the analysis on fractals to solve differential equations for real problems of fractal material is very limited due to weak development of this area of mathematics at this moment. Secondly, fractal materials and media cannot be described as fractals. The main property of the fractal is non-integer Hausdorff dimension [77] that should be observed on all scales. The fractal structure of real media cannot be observed on all scales from the infinitely small to the infinitely large sizes. Materials may have a fractal structure only for scales from the characteristic size of atoms or molecules of fractal media up to size of investigated sample of material.

The second approach, which is based on the use of fractional integration in integer dimensional spaces, can give adequate models to describe fractal media. The main disadvantage of the fractional–integral continuum models is the existence of various types of fractional integrals, which led to the arbitrariness in the choice of the correspondent densities of states.

In this paper, we consider the third approach used the non-integer dimensional spaces. One of the advantages of this approach is a possibility to avoid the arbitrary choice of densities of states. In addition, we also suggest a generalization