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Asymptotic properties and numerical simulation of multidimensional Lévy walks

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ABSTRACT

In this paper we analyze multidimensional Lévy walks with power-law dependence between waiting times and jumps. We obtain the detailed structure of the scaling limits of such multidimensional processes for all positive values of the power-law exponent. It appears that the scaling limit strongly depends on the value of the power-law exponent and has two possible scenarios: an α-stable Lévy motion subordinated to a strongly dependent inverse subordinator, or a Brownian motion subordinated to an independent inverse subordinator. Moreover, we derive the mean-squared displacement for the scaling limit processes. Based on these results we conclude that the resulting limiting processes belong to sub-, quasi- and superdiffusion regimes. The corresponding fractional diffusion equation and Langevin picture of considered models are also derived. Theoretical results are illustrated using the proposed numerical scheme for simulation of considered processes.

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1. Introduction

Starting with the pioneering work of Montroll & Weiss [1], continuous-time random walks (CTRWs) became one of the most popular and useful models in statistical physics. Their practical advantages were proved in spectacular way in the seminal paper [2], in which CTRW was applied as a model of charge carrier transport in amorphous semiconductors. Today CTRWs are well established mathematical models [3], which are particularly attractive in the description of anomalous dynamics [4]. They were used to model, among others, the following complex systems: dispersion in turbulent systems [5], plasma devices [6], tracer dispersion [7], electron transfer [8], gene regulation [9].

CTRW is a stochastic process determined uniquely by consecutive jumps of the random walker and waiting times between them. The jumps and the waiting times are drawn from some associated probability distributions. In the simplest setting, successive steps and times between them are assumed independent from the previous ones and from each other. Such uncoupled case with power-law distributions of jumps and waiting times leads to the space–time fractional Fokker–Planck equations [4,10,11] describing subdiffusion and Lévy flights [12–15]. Adding the dependence between consecutive waiting times results in correlated CTRWs, which have been recently investigated in [16–21].

One of the drawbacks of uncoupled CTRWs with heavy-tailed jumps is the diverging mean-square displacement (MSD). The remedy for this drawback can be found by introducing coupling (strong dependence) between jumps and rests of the walker. This way we arrive at the class of Lévy walks [22–24]. In the standard approach Lévy walk is a special type of CTRW, in which the length of each jump is equal to the length of the preceding waiting time (motion with constant velocity).

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Moreover, both jumps and rests are drawn from power-law distribution. Although, the jumps themselves can have infinite second moment, due to the coupling the MSD of the walker is finite (long jumps are penalized by long rests). There are many fascinating examples of applications of Lévy walks in the description of real-life phenomena: fluid flow in a rotating annulus [5], blinking nanocrystals [25], human travel [26,27], epidemic spreading [28,29], foraging of animals [30,31], transport of light in optical materials [32]. Some recent theoretical advances in the context of ergodicity breaking can be found in [33,34].

In this paper we proceed with further theoretical studies on Lévy walks. We introduce a generalized Lévy walk, in which the length of each d-dimensional jump is equal to the length of the corresponding waiting time raised to some power. Moreover, the direction of jump is governed by a random unit vector in $\mathbb{R}^d$. We consider both possible scenarios: wait-jump and jump-wait. The first case results in finite MSD with sub-, superdiffusive character (nonlinear MSD, see for instance [4,35]) or quasidiffusive character (linear MSD), depending on the power law parameters. However the second scenario leads to diverging second moment. We derive the asymptotic limit of the introduced walks and describe in detail the structure of the obtained subordinated processes. Moreover, we analyze the corresponding Langevin equations and governing fractional differential equations. We also describe the algorithm of simulating trajectories of the studied processes and present some numerical Monte Carlo results. Some recent results concerning scaling limits of standard Lévy walks can be found in [36,37]. General limit theorems for CTRWs can be found in [10,38–41].

We would like to underlined that there are two related, but different, definitions of Lévy walk in the literature. The first one, used in this paper, is based on the CTRW theory. The corresponding trajectories are discontinuous, since the walker performs jumps. The second definition assumes that the trajectories of the walker are continuous. They are constructed by linear interpolation of the trajectories of the underlying CTRW. The second case is not studied in this paper. Its analysis involves extension of the classical CTRW theory and will be the subject of our further studies.

This paper is structured in the following way: in Section 2 we introduce the definitions of multidimensional generalized Lévy walk (GLW) and generalized overshooting Lévy walk (GOLW) processes. In Section 3 we develop the limit theory of considered processes, derive their nonlinear MSD's, fractional equations as well as the Langevin representation. Section 4 presents approximation scheme for the scaling limits of GLW and GOLW. In Appendices A–C one can find proofs of the main results of the paper.

2. Generalized multidimensional Lévy walks

In this section we introduce the formal definitions of d-dimensional GLW and GOLW. Firstly let us recall the notion of a d-dimensional CTRW process.

Let $\{(T_i, \mathbf{J}_i)\}_{i\geqslant 1}$ be a sequence of independent and identically distributed (IID) random vectors such that $T_i \in \mathbb{R}_+$ and $\mathbf{J}_i \in \mathbb{R}^d$ denote consecutive waiting times and jumps of CTRW process, respectively. We assume that for each pair there is a possible coupling between T_i and $\mathbf{J}_i$. As a result the following random sum defines CTRW process [3]:

$$\mathbf{R}(t) = \sum_{i=1}^{N(t)} \mathbf{J}_i, \quad t \geqslant 0, \tag{1}$$

where $N(t)$ is a renewal process counting the number of jumps of the particle up to time t. It is given by

$$N(t) = \max\{n : T_1 + T_2 + \cdots + T_n \leqslant t\}.$$

A particle, which is driven by CTRW process, starts its motion at the origin and waits there for a random time T_1, then it performs the first jump $\mathbf{J}_1$ immediately. At the new position it waits for random time T_2 to performs jump $\mathbf{J}_2$ and then the scheme repeats.

Recently, we observe an increasing interest in certain modification of the classical CTRW, called overshooting continuous-time random walk (OCTRW) [40,42], defined as follows:

$$\tilde{\mathbf{R}}(t) = \sum_{i=1}^{N(t)+1} \mathbf{J}_i, \quad t \geqslant 0. \tag{2}$$

In comparison to the classical CTRW, where ith waiting time precedes ith jump, in the OCTRW scheme ith jump precedes ith waiting time (the particle first performs the jump and then waits). The OCTRW scheme plays an important role in the modeling of nonexponential relaxation patterns of dielectrics [40,42].

Next, let us assume that the waiting times are heavy-tailed distributed with index $\alpha \in (0,1)$, which means that the following property holds

$$\mathrm{P}(T_i > t) \propto t^{-\alpha}, \tag{3}$$

as $t \to \infty$. Additionally, we assume that the d-dimensional jumps of the walker are equal to

$$\mathbf{J}_i = v\mathbf{V}_i T_i^{\gamma}. \tag{4}$$

Here γ, v are positive real constants, $\{\mathbf{V}_i\}_{i\geqslant 1}$ is an IID sequence of non-degenerate unit vectors from $\mathbb{R}^d$ governing the direction of jumps. The sequences $\{\mathbf{V}_i\}_{i\geqslant 1}$ and $\{T_i\}_{i\geqslant 1}$ are assumed independent. Therefore, the length of ith jump is equal to the length of respective ith waiting time raised to power γ and multiplied by a factor v. The direction of ith jump is given by the random vector $\mathbf{V}_i$. In the following part we will assume for the simplicity that $v = 1$.

CTRW $\mathbf{R}(t)$ and OCTRW $\tilde{\mathbf{R}}(t)$ which fulfill conditions (3) and (4) are called GLW and GOLW processes, respectively.

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