Commun Nonlinear Sci Numer Simulat 19 (2014) 2220–2227

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Fractional differential equations solved by using Mellin transform

Salvatore Butera *, Mario Di Paola

Dipartimento di Ingegneria Civile Ambientale, Aerospaziale e dei Materiali (DICAM), Università degli Studi di Palermo, Viale delle Scienze, Ed. 8, 90128 Palermo, Italy

A R T I C L E I N F O

Article history:
Received 8 August 2013
Received in revised form 27 November 2013
Accepted 30 November 2013
Available online 11 December 2013

Keywords:
Fractional differential equations
Mellin transform
Self-similarity of inverse Mellin transform

A B S T R A C T

In this paper, the solution of the multi-order differential equations, by using Mellin transform, is proposed. It is shown that the problem related to the shift of the real part of the argument of the transformed function, arising when the Mellin integral operates on the fractional derivatives, may be overcame. Then, the solution may be found for any fractional differential equation involving multi-order fractional derivatives (or integrals). The solution is found in the Mellin domain, by solving a linear set of algebraic equations, whose inverse transform gives the solution of the fractional differential equation at hands.

1. Introduction

It is widely recognized that the memory and hereditary properties of various materials and processes in electrical circuits, biology, biomechanics, etc., such as viscoelasticity, is well predicted by using fractional differential operators. Such operators are the generalization, to real (or complex) order, of the classical derivatives and integrals (see e.g., [1–3]). Conversely to the locally defined classical derivatives, the powerful of the fractional operators in describing the time evolution of many physical processes and, in general, in modeling the dynamics of complex systems, is due to the long memory characteristics inherent to these operators. Indeed, dealing with systems characterized by a power type non local interaction, or by a non-Markovian power law time memory, into which the complexity of the dynamics usually manifest itself, fractional differential equations naturally arise in the relative mathematical models [4–6]. As a proof of their powerful in describing nature, theoretical research on these operators experienced an exceptional boost in the last few decades, and applications can now be found in various fields of natural sciences. Examples are in electrical circuits [7], in anomalous transport and diffusion processes in complex media [8–11], in material sciences [12–14], in biology [15–17] and biomechanics [18–20], and in many other branches of physics and engineering [21–23].

Various methods for the solution of differential equation of fractional order are available in literature, including Laplace method [21,24], Grünwald–Letnikov method [21,25], Adomian method [26] and several others [15,21,27–30]. Some attempts to use Mellin transform and related concepts, have been presented (see e.g., [21]) in order to solve particular classes of fractional differential equations.

In this paper, a general method of solution for Initial Value Problems (IVP), involving fractional derivatives, is presented by using the Mellin transform of complex order $\gamma = \rho + i\eta$. The method takes advantage of the fact that the discretized version of the inverse Mellin transform may be seen, in logarithmic temporal scale, as a Fourier series and, in time domain,

* Corresponding author. Address: SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom. Tel.: +39 3335779923.

E-mail addresses: sg.butera@gmail.com (S. Butera), mario.dipaola@unipa.it (M. Di Paola).

1007-5704/$ - see front matter
http://dx.doi.org/10.1016/j.cnsns.2013.11.022

as a complex Taylor series with coefficients depending on the fractional integrals in zero. The restitution of the function is independent on the value of ρ used to evaluate the discretized inverse Mellin transform, provided it belongs to its so called *fundamental strip*. With these informations in mind, in this paper, a method that allow us to relate the value of the Mellin transform for different values of ρ belonging to the fundamental strip, is presented. This is the main key to solve multi-order fractional differential equations in a very easy and direct way. The method is versatile and easy to implement in computer programs.

The paper is organized as follows: in the next section, the need to handle with fractional differential equation is presented, with a relevant example in which fractional differential equations appear and whose classical solution is written in terms of Mittag–Leffler series expansion; in Section 3, the Mellin transform and related concepts are highlighted; in Section 4, the solution of the fractional differential equation is presented, along with some applications in Section 5. In appendix some few basic elements on fractional calculus are reported for completeness sake's.

2. Fractional differential equations hereditariness

In this section, the relevant example of viscoelastic materials is presented in order to show the importance of the fractional calculus for many physical and engineering problems.

The linear viscoelastic problem is ruled by two different but interconnected functions: (i) the creep function, labelled as $J(t)$, that is the strain history for an imposed stress history $\sigma(t) = U(t)$ (unit step); (ii) the relaxation function, labelled ad $G(t)$, that is the stress history for an imposed strain history $\epsilon(t) = U(t)$. In linear viscoelasticity, the Boltzmann superposition principle holds, so that

$$\sigma(t) = \int_0^t G(t-\tau)\dot{\epsilon}(\tau)d\tau \tag{1a}$$

$$\epsilon(t) = \int_0^t J(t-\tau)\dot{\sigma}(\tau)d\tau \tag{1b}$$

Eq. (1), valid for quiescent systems in $t \leqslant 0$, suggest that relaxation and creep functions play the role of kernels, in (1a) and (1b) respectively. By using the Laplace transform of Eq. (1), the following fundamental relationship

$$\hat{J}(s)\hat{G}(s) = 1/s^2 \tag{2}$$

is easily derived, where $\hat{J}(s)$ and $\hat{G}(s)$ are the Laplace transform of $J(t)$ and $G(t)$ respectively. On the other hand, Nutting [31], by means of experimental tests performed on various materials like rubber, ceramics, etc., showed that in general the relaxation function may be written in the form

$$G(t) = \frac{c_\alpha}{\Gamma(1-\gamma)} t^{-\alpha} \quad (0 \leqslant \alpha \leqslant 1) \tag{3}$$

and, in virtue of Eq. (2), the corresponding creep function is

$$J(t) = \frac{1}{c_\alpha \Gamma(1+\gamma)} t^{\alpha} \quad (0 \leqslant \alpha \leqslant 1) \tag{4}$$

where $\Gamma(\cdot)$ is the Euler Gamma function, c_α and α are characteristic coefficients of the material at hands. As we insert Eqs. (3) and (4) in Eqs. (1a) and (1b), respectively we get

$$\sigma(t) = c_\alpha ({}^C\mathbf{D}^\alpha_{0^+}\epsilon)(t) \tag{5a}$$

$$\epsilon(t) = \frac{1}{c_\alpha}(\mathbf{I}^\alpha_{0^+}\sigma)(t) \tag{5b}$$

where $({}^C\mathbf{D}^\alpha_{0^+}\epsilon)(t)$ and $(\mathbf{I}^\alpha_{0^+}\sigma)(t)$ are the Caputo's functional derivative [32] and the Riemann–Liouville fractional integral, respectively (see Appendix A). From Eq. (5), some considerations may be drawn: (i) The viscoelastic constitutive law is ruled, in its direct and inverse form, by fractional operators (derivative and integral) of the same order (α). (ii) If $\alpha = 0$, then the elastic constitutive law is recovered while, if $\alpha = 1$, the Newton–Petrov constitutive law of pure fluid appears. It follows that the constitutive law of a viscoelastic material has an intermediate behavior between pure fluid and pure elastic solid. (iii) For quiescent systems for $t \leqslant 0$, the Caputo's fractional derivative coalesces with the Riemann–Liouville fractional derivative, and the two operators in Eqs. (5a) and (5b) are the inverse each another.

In order to capture different behaviors for more complex systems like bones [18], bitumen [33], and so on, the constitutive law has to be modified by inserting others fractional order operators, to obtain differential equations of the kind

$$\sigma(t) = \sum_{k=0}^{n} c_k^\alpha ({}^C\mathbf{D}^{\alpha_k}_{0^+}\epsilon)(t) \tag{6}$$