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Reaction diffusion equation with spatio-temporal delay ☆

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ABSTRACT

We investigate reaction–diffusion equation with spatio-temporal delays, the global existence, uniqueness and asymptotic behavior of solutions for which in relation to constant steady-state solution, included in the region of attraction of a stable steady solution. It is shown that if the delay reaction function satisfies some conditions and the system possesses a pair of upper and lower solutions then there exists a unique global solution. In terms of the maximal and minimal constant solutions of the corresponding steady-state problem, we get the asymptotic stability of reaction–diffusion equation with spatio-temporal delay. Applying this theory to Lotka–Volterra model with spatio-temporal delay, we get the global solution asymptotically tend to the steady-state problem's steady-state solution.

1. Introduction

In biology, many models can be attributed to delay reaction–diffusion equations, for example, epidemic model, population model, prey–predator model and so on [4,13,26]. In the past ten years, the theory of delay reaction–diffusion equations has attracted much attention, see [27].

The reaction–diffusion equations with a finite delay

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + f(u(x,t-\tau)), & x \in \Omega, \quad t \geqslant 0, \\ \mathcal{B}u = 0, \quad x \in \partial\Omega, \quad t \geqslant 0, \\ u(x,t) = \psi(x,t), \quad x \in \Omega, \quad -\tau < t \leqslant 0, \end{cases} \tag{1.1}$$

have been extensively studied, where $u \in \mathbf{R}$, Ω is a bounded region in $\mathbb{R}_N$ with smooth boundary $\partial\Omega$, $D = diag(d_1, d_2, \ldots, d_m)$ with $d_i > 0$ for $i = 1, 2, \ldots, m$, $\mathcal{B}$ is the Dirichlet or Neumann boundary operator. The study of Eq. (1.1) was initiated by Travis and Webb [21,22], Webb [24,25], Fitzgibbon [5] and Rankin [19] for the existence, stability and asymptotic behavior of solutions. For later development, we cite Martin and Smith [11–13], Wu [27], Berres and Ruiz-Baier[2] and so on.

It should be mentioned that the reaction–diffusion equation with infinite delay also have been investigated in [7,28]. Motivated by Martin and Smith [11–13], Hale and Kato [6], Ruan and Wu [20] developed a general theory of existence, comparison, invariance, monotonicity and set-condensing for partial functional differential equations with infinite delay and provided some applications in reaction–diffusion systems with general distributed delays. For recent development, we can refer to Pao [15–18].

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On the other hand, in many practical problems individuals usually move around. In this case, it is not sufficient to include only a time delay in models. For example, Britton [3,4] introduced spatio-temporal delay or nonlocal delay. Afterwards, lots of work has been done on reaction–diffusion equations with spatio-temporal delay, we refer to [8,10,14,23]. It is noteworthy that these research works focused on the traveling wave solutions of reaction–diffusion equation with spatio-temporal delay. From another point of view, we study the initial value problem of reaction–diffusion equations with spatio-temporal delay.

In this paper, we will mainly study the following reaction–diffusion equation with spatio-temporal delay.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma u(x,t) + f(u,(x,t), g * h(u(x,t))), \quad t \leqslant 0, \quad x \in \mathbb{R} & (1.2) \\ u(x,t) = \psi(x,t), \quad t \leqslant 0, \quad x \in \mathbb{R} & (1.3) \end{cases}$$

where $D > 0$ is the diffusion rate, $\gamma > 0$ is the per capita mortality rate of the species at location x, the convolution $g * h(u(x,t))$ is defined by

$$g * h(u(x,t)) = \int_0^\infty \int_{-\infty}^\infty G(s,x,y)k(s)h(u(y,t-s))dyds,$$

with $h(u)$ being a continuous function, $k(t)$ a probability density function satisfying

$$\begin{aligned} & k(t) \geqslant 0, \quad \text{for all} \quad t \geqslant 0, \\ & \int_0^\infty k(t)dt = 1 \end{aligned} \tag{1.4}$$

and

$$G(t,x,y) = \frac{1}{\sqrt{4D\pi t}} \exp\left(-\frac{(x-y)^2}{4Dt}\right). \tag{1.5}$$

The initial data ψ is a given continuous mapping from $\mathbb{R} \times (-\infty, 0]$ into $\mathbb{R}$.

In fact, various types of equations can be derived from system (1.2) by taking different kernels. If $G(t,x,y) = \delta(x-y), k(t) = \delta(t-\tau)$, where δ is Dirac's delta function, $\tau > 0$ is a constant, then (1.2) changes into the following form

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma u(x,t) + \bar{f}(u(x,t), u(x,t-\tau)), \quad t \geqslant 0, \quad x \in \mathbb{R}, \tag{1.6}$$

where $\bar{f}(x,y) = f(x,h(y))$. Eq. (1.6) has been studied by several investigators, see, e.g., [27].

If $G(t,x,y) = \delta(x-y), k(t) = \delta(t)$, then (1.2) becomes the following reaction–diffusion equation without delay

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma u(x,t) + \tilde{f}(u(x,t)), \quad t \geqslant 0, \quad x \in \mathbb{R},$$

where $\tilde{f}(x) = f(x,h(x))$.

If $G(t,x,y) = \delta(x-y)$, then (1.2) changes into the following equation with distributed delay

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma u(x,t) + f(u(x,t), \int_{-\infty}^t k(t-s)h(u(x,s)ds)), \quad t \geqslant 0, \quad x \in \mathbb{R}.$$

The purpose of this paper is twofold. (I) Show the existence and uniqueness of a global solution for system (1.2) and (1.3). And (II) investigate the dynamic property of (1.2) and (1.3) by establishing a global attractor in the relation to the maximal and minimal constant solutions of the corresponding steady-state system. The existence and uniqueness of global solution are given in Section 2, while the dynamic problem is discussed in Section 3. In Section 4, we give an application of our results obtained to Lotka–Volterra model.

2. Existence theorems of the system with spatio-temporal delay

This section is motivated by Ruan and Wu [20], Wen and Xu [26]. The purpose of this section is to tackle the existence and uniqueness of global solution for problem (1.2) and (1.3).

If $f(x,y) = y$, then (1.2) and (1.3) turns to the system that Wen and Xu [26] have studied. Thus, the existence-uniqueness theory of (1.2) and (1.3) is more general than that in [26].

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$. Then there exists a closed cone

$$X_+ = \{u \in X; u(x) \geqslant 0, x \in \mathbb{R}\} \subset X,$$

which induces a partial ordering $\geqslant_X$ on X defined by

$$u \geqslant_X v, \quad \text{iff } u - v \in X_+.$$