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An explicit recursive formula for computing the normal forms associated with semisimple cases

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ABSTRACT

This paper presents an explicit, computationally efficient, recursive formula for computing the normal forms, center manifolds and nonlinear transformations for general n-dimensional systems, associated with semisimple singularities. Based on the formula, we develop a Maple program, which is very convenient for an end-user who only needs to prepare an input file and then execute the program to "automatically" generate the result. Several examples are presented to demonstrate the computational efficiency of the algorithm.

1. Introduction

Normal form theory has been used for several decades as one of the important tools in simplifying the study of nonlinear differential systems. Its basic idea is to introduce a near-identity transformation into a given differential system to eliminate as many of the nonlinear terms as possible, which are usually called non-resonant terms. The terms retained in the resulting system are normal form terms, called resonant terms. Since normal forms keep the fundamental dynamical characteristics of the original system in the vicinity of a singular point, it can be used to study the local bifurcations and stability/instability properties of the original system. There are various of books which have extensive discussions on normal form theory, for example, see [1–3]. More recent progress can be found in the article [4].

For higher-dimensional dynamical systems, normal form theory is usually applied together with center manifold theory, see [5–9]. If the Jacobian matrix of a differential system evaluated at a singular point contains eigenvalues with zero real part and non-zero real part, then center manifold theory should be considered in the study of the local dynamics of the system, and the dimension of the center manifold is equal to the number of eigenvalues with zero real part. Center manifold theory plays an important role in simplifying the analysis of local dynamical behavior of nonlinear differential systems near a singular point, because it allows us to determine the behavior by study the flow on a lower dimensional manifold.

Several computer algebra systems such as Maple, Mathematica, Macsyma, etc., have been widely used for the computation of normal forms. Even with the help of these computer algebra systems, it is still not easy to obtain higher-order normal forms since considerably more computer memory and computational time are demanded as the order of normal forms increases. Therefore, in the past two decades, various methods have been developed to compute normal forms for general n-dimensional differential systems. However, many methods are not computationally efficient because lots of unnecessary computations are involved, for example, see [6,10,11]. To be precise, in order to get an expression for the kth-order normal

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form computation, $(k-1)$th-order normal forms, center manifolds and near-identity transformation are substituted into the original system. Thus, besides the kth-order terms, the obtained expression also contains lower-order ($<k$) and higher-order ($>k$) terms, which are not desirable for efficient computation. To overcome this problem, Yu [7,12] developed a recursive formula for computing the coefficients of normal forms and center manifolds, which avoid those lower-order ($<k$) and higher-order ($>k$) terms in the kth-order computation. However, these formulas are not given in explicit recursive expressions and may be not so efficient in computation. For general planar systems, [13] obtained an explicit recursive formula for computing Poincaré–Lyapunov constants (focus values), and the computation based on this formula is efficient.

In this paper, we consider general n-dimensional differential systems associated with semisimple cases, i.e., the Jacobian matrix of the linearized system evaluated at a singular point can be transformed into a diagonal Jordan canonical form. Around semisimple singularities, a rich variety of bifurcations, such as Hopf, double-zero, Hopf-zero, double-Hopf, etc. may occur. A detailed study for some types of these bifurcations can be found in [14, chap. 7] by applying normal form theory to simplifying the systems. Particularly, for some special bifurcations like Hopf-zero, double-Hopf without resonance, the normal forms are symmetric with respect to rotation in the direction associated with the imaginary eigenvalues. In this case, the normal forms can be decoupled, and the systems are further simplified. Many methods have been developed and used to compute the normal forms of systems with semisimple singularities, not only for the particular cases like Hopf [9,12,13], Hopf-zero [15] and double-Hopf [16,17], but also for general semisimple cases involving center manifold [6,7]. In order to provide a good algorithm to compute the normal forms of general cases, in this paper we will develop a computationally efficient method and a Maple program without restriction on the dimension of the center manifold. This paper is an extension of our recent work [9], which focuses on general differential systems associated with Hopf bifurcation.

In the next section, an explicit, computationally efficient, recursive formula is derived for computing the normal forms and center manifolds of dynamical systems associated with semisimple singularities. The explicit formula is given in terms of the system coefficients of the original differential system, which is easily used for developing a Maple program. In Section 3, several examples are presented to demonstrate the computational efficiency of the method and the Maple program. Finally, conclusion is drawn in Section 4.

2. Main result

Consider a system of differential equations in the general form,

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{G}(\mathbf{y}), \quad \mathbf{y} \in \mathbf{R}^n, \quad \mathbf{G}(\mathbf{y}) : \mathbf{R}^n \to \mathbf{R}^n, \tag{1}$$

where the dot represents differentiation with respect to time, t, the matrix A is diagonalizable, $\mathbf{G}(\mathbf{0}) = \mathbf{0}$ and $D_{\mathbf{y}}\mathbf{G}(\mathbf{0}) = \mathbf{0}$. Denote by $\lambda_i, i = 1, \ldots, n$, the eigenvalues of A. Without loss of generality, it is assumed that there are only k eigenvalues $\lambda_j, j = 1, \ldots, k$, having zero real part, implying that system (1) has a k-dimensional center manifold.

Then, through a proper linear transformation, system (1) can be transformed into

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{f}(\mathbf{x}), \tag{2}$$

where J is a diagonal matrix, and $\mathbf{f}(\mathbf{x})$ is expanded as

$$\mathbf{f}(\mathbf{x}) = \sum_{m \geqslant 2} \mathbf{f}_m(\mathbf{x}), \quad \text{where } \mathbf{f}_m(\mathbf{x}) = \sum_{\{m(n)\}} \mathbf{f}_{m(n)} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

and $m(n)$ denotes a vector $(m_1, m_2, \ldots, m_n)$ of n nonnegative integers, which satisfies $\sum_{j=1}^{n} m_j = m$.

Suppose that the matrix J has the form $J = \mathrm{diag}(J_o, J_r)$, where

$$J_o = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \quad J_r = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n).$$

Let $\mathbf{x} = (\mathbf{x}_o^T, \mathbf{x}_r^T)^T$, where $\mathbf{x}_o = (x_1, x_2, \ldots, x_k)^T$ and $\mathbf{x}_r = (x_{k+1}, x_{k+2}, \ldots, x_n)^T$. Then, system (2) can be written as

$$\begin{aligned} \dot{\mathbf{x}}_o &= J_o\mathbf{x}_o + \mathbf{f}_o(\mathbf{x}_o, \mathbf{x}_r), \\ \dot{\mathbf{x}}_r &= J_r\mathbf{x}_r + \mathbf{f}_r(\mathbf{x}_o, \mathbf{x}_r). \end{aligned} \tag{3}$$

The center manifold of (3) may be defined as $\mathbf{x}_r = \mathbf{H}(\mathbf{x}_o)$, which satisfies $\mathbf{H}(\mathbf{0}) = \mathbf{0}$, $D\mathbf{H}(\mathbf{0}) = \mathbf{0}$. Then, the differential equation describing the dynamics on the center manifold is given by

$$\dot{\mathbf{x}}_o = J_o\mathbf{x}_o + \mathbf{f}_o(\mathbf{x}_o, \mathbf{H}(\mathbf{x}_o)). \tag{4}$$

Next, introduce a near-identity nonlinear transformation, given by

$$\mathbf{x}_o = \mathbf{u} + \mathbf{Q}(\mathbf{u}) = \mathbf{u} + \sum_{m \geqslant 2} \sum_{\{m(k)\}} \mathbf{q}_{m(k)} u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k} \equiv \mathbf{q}(\mathbf{u}), \tag{5}$$

into (4) to obtain the normal form,

$$\dot{\mathbf{u}} = J_o\mathbf{u} + \mathbf{C}(\mathbf{u}), \quad \text{where } \mathbf{C}(\mathbf{u}) = \sum_{m \geqslant 2} \sum_{\{m(k)\}} \mathbf{c}_{m(k)} u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k}. \tag{6}$$