Commun Nonlinear Sci Numer Simulat 19 (2014) 2493–2504

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Bipartite and directed scale-free complex networks arising from zeta functions

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ARTICLE INFO

Article history:
Received 25 July 2012
Received in revised form 5 March 2013
Accepted 30 August 2013
Available online 8 September 2013

Keywords:
Complex networks
Bipartite graphs
Directed graphs
Multiple zeta functions

ABSTRACT

We construct a new class of directed and bipartite random graphs whose topology is governed by the analytic properties of multiple zeta functions. The bipartite *L*-graphs and the multiplicative zeta graphs are relevant examples of the proposed construction. Phase transitions and percolation thresholds for our models are determined.

1. Introduction

In the last decade, complex networks have been acquiring a prominent role in different branches of science as theoretical physics, biology, information science, social sciences, etc. (see, e.g., [28], and the reviews [2,7,9,13,14,27]). Indeed, they are essential in modeling systems with nontrivial interactions, and are usually represented in terms of *random graphs* [8,21]. Phenomena like phase transitions in complex networks depend crucially on the topology of the underlying graphs.

Since the pioneering work of Erdös and Rényi [16], and Solomonoff and Rapoport [33], this field has known a dramatic development, and has been widely investigated [8]. Many new models of random graphs have been considered and their role in the applications analyzed.

In particular, *scale-free* models, i.e. models exhibiting a power-law degree distribution, represent one of the most studied classes of complex networks. Historically, the first example of them was offered by the Price model [32]. Among the most important ones are those proposed by Barabasi, Albert and collaborators in [5,6] and by Aiello et al. in [1] (for a recent review, see the monograph [9]).

The aim of this work is to establish a connection between the theory of complex networks and number theory. This research can be considered part of a general program aiming at investigating the relation between statistical mechanics and number theory. A first connection between these two fields was discovered by Montgomery and Odlyzko: the Gaussian unitary ensemble was related with the zeros of the Riemann zeta function $\zeta(s)$. Since then, many studies have been devoted to clarify the relation among generalized zeta functions, random matrix theory, and various aspects of quantum field theory and spectral theory (see [11,23] for general reviews).

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http://dx.doi.org/10.1016/j.cnsns.2013.08.037

In [37], L-functions, which are among the most important objects in analytical number theory, have been used to construct scale-free graphs possessing several interesting topological properties. Essentially, L-functions are meromorphic continuations of Dirichlet series to the whole complex plane, with a Euler product and a functional equation (for a modern introduction, see e.g. the monograph [20]). The classical Riemann zeta function is the most known example of a L-function. An axiomatic theory of L-functions has been proposed by Selberg [22]. In [36], Dirichlet series have been related to generalized entropies via the notion of universal formal group (see also [34,35].)

In this paper, we further extend the results of [37] and construct new families of *directed and bipartite random graphs with scale similarity properties*. The main motivation is that, apart their intrinsic theoretical interest, graphs of this class appear in many real-world networks.

A directed graph is a graph in which a direction is specified in every link. This makes directed graphs more sophisticated and realistic than undirected graphs. A basic example of directed graph is the world-wide web.

In turn, bipartite graphs are characterized by two different types of vertices, each endowed with a degree distribution. These models are also widely investigated for their usefulness in different applications ([2,28]).

As is common in the literature, the scale-free invariance of a unipartite model essentially means that, given the probability distribution $\{p_k\}_{k\in\mathbb{N}}$, the ratio $p_{\alpha\times k}/p_k$ depends only on α but not on k. In the following, we will consider a natural generalization of this notion for classes of directed and bipartite random graphs.

We are here mainly interested in the topology of the new networks we introduce. In particular, we will study the conditions under which phase transitions may occur. These condition will be typically expressed in terms of functional equations in the parameters of our models, that can be solved numerically with arbitrary accuracy.

The paper has the following structure. In Section 2, some basic definitions concerning the theory of generating functions for random graphs are proposed. In Section 3, bipartite graphs arising from L-functions are introduced. In Section 4, an analogous construction is proposed for the case of directed graphs. In Section 5, the relevant subcase of scale-free networks is analyzed in detail, in terms of a suitable group theoretical structure allowing the composition of graphs. In Section 6, an application of our models to a biological context is proposed.

2. Algebraic preliminaries

In order to fix the language and the notation, in this Section some basic aspects of the formalism adopted in the paper will be sketched.

Throughout this work, we will stay in the limit of large graph size.

In order to define a random graph, a degree probability distribution $\{p_n\}_{n\in\mathbb{N}}$ of vertices in the graph is introduced, where p_n is the probability that an uniformly randomly chosen vertex has degree n. Once assigned a degree probability distribution, a graph is chosen uniformly at random in the class of all graphs with that given distribution.

Consider first the case of an *undirected graph*. The series

$$G_0(x) := \sum_{n=0}^{\infty} p_n x^n, \tag{1}$$

is called the generating function of the distribution. We have necessarily

$$G_0(1) = 1. \tag{2}$$

The distribution of the outgoing edges is generated by

$$G_1(x) := \frac{\sum_{n=1} n p_n x^{n-1}}{\sum_{n=1} n p_n} = \frac{1}{< m >} G_0'(x), \tag{3}$$

where the average number of first neighbors, equal to the average degree of the graph, is

$$z_1 =< n >= \sum_n n p_n = G_0'(1).$$

In the case of *directed graphs*, each vertex possesses an in-degree j and an out-degree k. Therefore, one introduces a distribution $\{\pi_{jk}\}_{j,k\in\mathbb{N}}$ over both degrees. The generating function for a directed graph is of the form

$$G(x,y) = \sum_{j,k} \pi_{jk} x^j y^k. \tag{4}$$

It is natural to introduce generating functions for the in-degrees and out-degrees, which are obtained from Eq. (4) by summing away the irrelevant degrees of freedom:

$$F_0(x) = G(x,1); \quad G_0(y) = G(1,y). \tag{5}$$