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Electrical analogous in viscoelasticity

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ABSTRACT

In this paper, electrical analogous models of fractional hereditary materials are introduced. Based on recent works by the authors, mechanical models of materials viscoelasticity behavior are firstly approached by using fractional mathematical operators. Viscoelastic models have elastic and viscous components which are obtained by combining springs and dashpots. Various arrangements of these elements can be used, and all of these viscoelastic models can be equivalently modeled as electrical circuits, where the spring and dashpot are analogous to the capacitance and resistance, respectively. The proposed models are validated by using modal analysis. Moreover, a comparison with numerical experiments based on finite difference time domain method shows that, for long time simulations, the correct time behavior can be obtained only with modal analysis. The use of electrical analogous in viscoelasticity can better reveal the real behavior of fractional hereditary materials.

1. Introduction

In the last few decades, fractional calculus has attracted a great interest in various scientific areas including physics and engineering [1–9]. Particularly, in the area of viscoelasticity a significant effort has been done in describing more closely the behavior of materials by using fractional mathematical models. Moreover, the analogy between viscoelastic and electrical constitutive equations is well-known so that, in spite of different physical meanings, the widely used Maxwell model, Kelvin–Voigt model, and Standard Linear Solid Model can be applied to predict a circuit behavior as well [10]. Besides, allow for the time varying distributions of elements, a series of generalized models are proposed in either canonical structure or ladder networks [11,12], such as the Maxwell–Wiechert model. All the above mentioned viscoelastic models have elastic and viscous components which are combined of springs and dashpots. The only difference is the arrangement of these elements, and all of these viscoelastic models can be equivalently modeled as electrical circuits, where the spring and dashpot are analogous to the capacitance and resistance respectively [13–15]. Nevertheless, compare to two viscoelastic elements, there are four passive electrical elements including resistor, capacitor, inductor and the recently find memristor [16,17]. Thus, although the circuits of LC, RC, RL, etc. can be transformed in some circumstances, it is still reasonable to expect that there are far more new properties included in the electrical models that are formulated by using the same structure in viscoelastic

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models. Particularly, the introduction of the fractional elements [18] and power-law phenomena cannot only extend the above discussions but also better reveal the real physical world such as the mechanical model of fractional hereditary materials [19] and the Abel's singular problem [20]. In the paper, mechanical models of viscoelasticity behavior are firstly approached by using fractional operators, based on recent works by the authors [19,21,22]. Then, electrical analogous models are introduced in order to obtain electrical equivalent circuits useful to predict the behavior of fractional hereditary materials in an easy way. The validity of the proposed models is demonstrated by using modal analysis. Moreover, the comparison with numerical experiments based on finite difference time domain (FDTD) method shows that, for long time simulations, the correct time behavior can be obtained only with modal analysis.

2. Mechanical models of fractional viscoelasticity

Many materials, like rubbers, polymers, bones, bitumen and so on, show a viscoelastic mechanical behavior; moreover also biological tissues have viscoelastic properties [23–28]. Viscoelasticity is the property of such materials that exhibit at the same time elastic and viscous behavior. The elastic behavior is typical of simple solid materials in which the strain history $\gamma(t)$ is linked by the stress history $\sigma(t)$ through a proportional relation as shown in Eq. (1):

$$\sigma(t) = E\gamma(t) \tag{1}$$

where E is the Young modulus (Pascal). Eq. (1) shows the so-called Hooke law, the mechanical model of elasticity is represented by a perfect spring with stiffness E as shown in Fig. 1(a). Instead, the viscous behavior is typical of perfect fluid in which there are stress and strain history linked by the Newtonian law as shown in following equation:

$$\sigma(t) = \eta \frac{d}{dt}\gamma(t) = \eta\dot{\gamma}(t) \tag{2}$$

where η is the viscosity (Poise) of the fluid. In this case the stress history $\sigma(t)$ is related to the rate of deformation $\dot{\gamma}(t)$ and the model that describes this behavior is the dashpot shown in Fig. 1(b).

In order to describe the viscoelastic behavior, the mechanical models of Fig. 1 are inadequate and, over time, some researchers have used several more or less complex assemblies of the two simple elements of Fig. 1, as it was done by Kelvin, Voigt, Maxwell, Zener, etc. [29,30]. In these models the stress–strain relation is described by following relation:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}\sigma(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k}\gamma(t) \tag{3}$$

Another way to describe the time dependent behavior of viscoelastic materials is by the integral formulation. In fact, from the relaxation function $G(t)$, that represents the stress $\sigma(t)$ for an assigned strain history $\gamma(t) = H(t)$ (where $H(t)$ is the unit step function) and by using the Boltzmann superposition integral the following stress–strain relation is obtained:

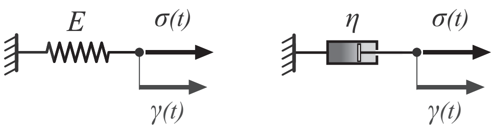
$$\sigma(t) = \int_0^t G(t-\bar{t})d\gamma(\bar{t}) = \int_0^t G(t-\bar{t})\dot{\gamma}(\bar{t})d\bar{t} \tag{4}$$

Eq. (4) is valid for quiescent system at $t = 0$. All classical models, whose constitutive law is described by Eq. (3), have as relaxation function $G(t)$ a function based on an exponential law. Several scientists have experimentally demonstrated that the relaxation function is not well described by an exponential law, but it follows a power-law trend [21,31–34] of the following type:

$$G(t) = \frac{C(\beta)}{\Gamma(1-\beta)} t^{-\beta} \tag{5}$$

where $\Gamma(\cdot)$ is the Euler gamma function, $C(\beta)$ and β are parameters that depend on the specific material. By using the relaxation function of Eq. (5) and applying the Boltzmann superposition integral of Eq. (4), another stress–strain relation is obtained:

$$\sigma(t) = C(\beta)\left({}_C D_{0^+}^{\beta}\gamma\right)(t) \tag{6}$$



(a) Perfect spring (b) Perfect dashpot

Fig. 1. Elastic and viscous models.