Commun Nonlinear Sci Numer Simulat 19 (2014) 1651–1660

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Global exponential stability for coupled retarded systems on networks: A graph-theoretic approach

Wenxue Li *, Hongwei Yang, Liang Wen, Ke Wang

Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai 264209, PR China

ARTICLE INFO

Article history:
Received 30 November 2012
Received in revised form 30 September 2013
Accepted 30 September 2013
Available online 9 October 2013

Keywords:
Coupled retarded systems
Networks
Exponential stability
Razumikhin method

ABSTRACT

In this paper, some coupled retarded systems on networks (CRSNs) are studied. Applying the idea of Razumikhin method and Mirchhoff's matrix tree theorem, the sufficient conditions for global exponential stability of the CRSNs are obtained. Finally, an example of coupled retarded oscillator system on networks is given to illustrate the advantages of our results.

1. Introduction

In recent years, coupled systems on networks (CSNs) have been widely studied owning to their potential applications in the areas of biological systems [1,2], neural networks [3,4], epidemic models [5,6] and chemical systems [7–9]. At the same time, the stability for delay neural networks has attracted the attention of an increasing number of scientists due to their potential application in different areas such as applications of synchronization of coupled oscillators [10–12] and consensus problems of networked control systems [13]. The stability of equilibrium is the precondition of applications of CSNs in practice. Therefore, there are a great amount of literatures on this topic. In [9,14], Li et al. used graph theory to explore the global stability for general coupled systems of ordinary differential equations on networks. Furthermore, by using the results, some novel sufficient conditions of global stability for some mathematical models were effectively given in [5,6,15–22]. In [23], Chen and Sun investigated the coupled systems on networks with constant delay by applying the technique in [9]. Moreover, the method was extended to stochastic cases in [8,24], and to discrete time neural networks with finite and infinite delays in [3].

In the study of coupled system, it is an interested problem that the change of connections affects the dynamical properties of coupled system. In [25], Achlioptas et al. explored that incorporating a limited amount of choice in the classic Erdö–Rényi network formation model causes its percolation transition to become discontinuous. In [26], Sakyte and Ragulskis investigated that two stable regimes of the complete network can coexist under continuous weak stimulation: the oscillatory synchronized regime and the quiet regime, where all neurons stop firing completely.

Because the dynamics of coupled systems depends on not only the individual vertex dynamics but also the coupling topology, the stability analysis for coupled retarded systems on networks (CRSNs) is generally a complex and formidable task. In this paper, we use the technique in [9,14] to consider the exponential stability of CRSN. In reference to the results, our contributions are as follows:

* Corresponding author. Tel.: +86 0631 5687035; fax: +86 0631 5687572.
E-mail addresses: wenxuetg@hitwh.edu.cn, wenxue810823@163.com (W. Li).

1007-5704/$ - see front matter
http://dx.doi.org/10.1016/j.cnsns.2013.09.039

- Both Razumikhin method and Mirchhoff's matrix tree theorem are used to obtain the sufficient conditions for the global exponential stability of CRSN (1). The novel approach that allows one to construct Lyapunov functions for CRSN (1) by using Lyapunov functions of vertex systems in the network.
- These conditions obtained in this paper are presented in terms of the topological property of networks and have an important leading significance in the design and applications of globally exponentially stable CRSNs.

The organization of this paper is as follows. The problem formulation and some basic preliminaries are given in Section 2. In Section 3, the main results, which guarantee the CRSN is globally stable, are provided. In Section 4, we discuss a numerical example to illustrate the advantages of our results.

2. Preliminaries and model formulation

In this section, we start with some useful notations for the simplicity. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbb{R}^1_+ = [0, +\infty), \mathbb{Z}^+ = \{1, 2, \cdots\}, \mathbb{L} = \{1, 2, \cdots, l\}$ and $m = \sum_{i=1}^{l} m_i$ for $m_i \in \mathbb{Z}^+$. Let $|\cdot|$ be the Euclidean norm for vectors or the trace norm for matrices. Let $C([-\tau, 0]; \mathbb{R}^n), \tau > 0$ be the space of continuous functions $x : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|x\| = \sup_{-\tau \leqslant u \leqslant 0} |x(u)|$.

The following basic concepts and theorem on graph theory can be found in Refs. [9,27]. A digraph $\mathcal{G} = (U, E)$ contains a set $U = \{1, 2, \cdots, l\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight a_{ij}. Here $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$, and we call $A = (a_{ij})_{l \times l}$ as the weight matrix. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \cdots, i_s\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \cdots, s-1\}$. If $i_s = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Denote the digraph with weight matrix A as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as $L = (p_{kh})_{l \times l}$, where $p_{kh} = -a_{kh}$ for $k \neq h$ and $p_{kh} = \sum_{j \neq k} a_{kj}$ for $k = h$.

Lemma 1 [9]. *Assume $l \geqslant 2$ and c_k denotes the cofactor of the k-th diagonal element of Laplacian matrix of $(\mathcal{G}, A)$. Then the following identity holds: $\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(x_k, x_h) = \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(k,h) \in E(C_{\mathcal{Q}})} F_{hk}(x_h, x_k)$. Here $F_{kh}(x_k, x_h)$ is an arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$, and $E(C_{\mathcal{Q}})$ is the set of arcs in $C_{\mathcal{Q}}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for $1 \leqslant k \leqslant l$.*

In this paper, we consider the global exponential stability for the following CRSNs:

$$\dot{x}^{(k)}(t) = f_k\big(x^{(k)}(t), x^{(k)}(t - \tau_k), t\big) + \sum_{h=1}^{l} H_{kh}\big(x^{(h)}(t - \tau_h)\big), \quad k \in \mathbb{L}, \ t \geqslant 0, \tag{1}$$

where $\tau_k \geqslant 0, x^{(k)}(t) \in \mathbb{R}^{m}_{k}, f_k : \mathbb{R}^{m}_{k} \times \mathbb{R}^{m}_{k} \times \mathbb{R}^1_+ \to \mathbb{R}^{m}_{k}$ and $H_{kh} : \mathbb{R}^{m}_{h} \to \mathbb{R}^{m}_{k}$ are continuous functions and satisfy $f_k(0, 0, t) = 0$ and $H_{kh}(0) = 0$ which implies that CRSN (1) has equilibrium $x^* = (x_1^*, x_2^*, \cdots, x_l^*)^T = 0$. Here, we assume that functions f_k and H_{kh} for $k, h \in \mathbb{L}$ satisfy Lipschitz condition:

1. For each $k \in \mathbb{L}$, there is a positive constant L_k such that

$$|f_k(x_k, y_k, t) - f_k(\bar{x}_k, \bar{y}_k, t)| \leqslant L_k(|x_k - \bar{x}_k| + |y_k - \bar{y}_k|)$$

for all $x_k, y_k, \bar{x}_k, \bar{y}_k \in \mathbb{R}^{m}_{k}$ and $t \geqslant 0$.

2. For each $k, h \in \mathbb{L}$, there is a positive constant $\bar{L}_{kh}$ such that

$$|H_{kh}(z_k) - H_{kh}(\bar{z}_k)| \leqslant \bar{L}_{kh}|z_k - \bar{z}_k|$$

for all $z_k, \bar{z}_k \in \mathbb{R}^{m}_{h}$ and $t \geqslant 0$.

By Theorem 2.2.3 in [28], CRSN (1) has a unique solution $x(t) = (x^{(1)}(t), x^{(2)}(t), \cdots, x^{(l)}(t))^T$ with initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0],$$

where $\phi(t) = (\phi^{(1)}(t), \ldots, \phi^{(l)}(t))^T$ is a continuous vector function on $[-\tau, 0]$, in which $\tau = \max\{\tau_1, \cdots, \tau_l\}$.

A digraph $\mathcal{G}$ with l vertices can be constructed for (1) as follows: each vertex represents a subsystem and the dynamics is defined by retarded differential system

$$\dot{x}^{(k)}(t) = f_k\big(x^{(k)}(t), x^{(k)}(t - \tau_k), t\big), \quad t \geqslant 0. \tag{2}$$