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Existence and uniqueness of solutions of initial value problems for nonlinear langevin equation involving two fractional orders ☆

Tao Yu, Ke Deng, Maokang Luo *

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

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The solvability of initial value problems for nonlinear Langevin equation involving two fractional orders are discussed in this paper. An existence result for the solution is obtained using the Leray–Schauder nonlinear alternative. In addition, sufficient conditions for unique solution are established under the Banach contraction principle. The existence results for the initial value problems of nonlinear classical Langevin equation follow as a special case of our results.

1. Introduction

The Langevin equation was proposed by Langevin [1] in 1908 to give an elaborate description of Brownian motion. In his work, Newton's second law was applied to a Brownian particle to invent the "F = ma" of stochastic physics which is now called "Langevin equation". On the other hand, Einstein'method of studying Brownian motion is based on the Fokker–Planck equation governing the time evolution of the Brownian particle's probability density. Langevin's approach is more simple than Einstein's at the cost of forcing into existence new mathematical objects (Gaussian white noise and the stochastic differential equation) with unusual properties. For a long time, the Langevin equation was widely used to describe the dynamical processes taken place in fluctuating environments [2]. However, for systems in disordered or fractal medium, some interesting phenomena such as anomalous transport [3] are observed. In these cases, ordinary Langevin equation cannot give a correct description of the dynamics any more. Thus, generalized Langevin equation (GLE) was introduced by Kubo [4] in 1966, where a frictional memory kernel was incorporated into the Langevin equation to describe the fractal and memory properties. The generalization of Langevin equation has since become a hot research topic.

As the intensive development of fractional derivative, a nature generalization of Langevin equation is to replace the ordinary derivative by a fractional derivative to yield fractional Langevin equation (FLE), which can be considered as a particular case of the GLE. FLE was introduced by Mainardi and collaborators [5,6] in earlier 1990s. The literature on this respect is huge, several different types of FLE were studied in [7–14]. The usual FLE involving only one fractional order was studied in [7,8]; the Langevin equation containing both frictional memory kernel and fractional derivative was studied in [9,10]; the

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* Corresponding author. Tel.: +86 13982254736.
E-mail addresses: scuyutao@163.com (T. Yu), dk_83@126.com (K. Deng), makaluo@scu.edu.cn (M. Luo).

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nonlinear Langevin equation involving two fractional orders was studied in [11–14]. We focus on the last type of FLE proposed first by Lim et al. [11] in 2008:

$$^{c}{}_{0}\mathcal{D}_t^{\beta}(^{c}{}_{0}\mathcal{D}_t^{\alpha} + \gamma)x(t) = f(t, x(t))$$

Compared with the usual FLE involving only one fractional order, the solution to this new version of FLE gives a fractional Gaussian process parametrized by two indices, which is more flexible for modeling fractal processes. Actually, if $\alpha + \beta = 2$ and $x(t)$ is sufficiently smooth, this new version of FLE will degenerate to the usual FLE considered in [7,8].

As we know, the linear fractional differential equation can always be analytically solved by means of the Laplace transform method [15]. Unfortunately, most of the nonlinear fractional differential equation can only be studied through numerical simulation. In order to ensure the reliability of simulation results, the existence and uniqueness of the solutions should be verified in the first place. Recently, the existence and uniqueness of solutions of the initial and boundary value problems for nonlinear fractional equations are extensively studied. We only mentation here the papers of Kosmatov et al. [16], Deng et al. [17], Agarwal et al. [18], Zhao et al. [19], Balachandran et al. [20] and the reference therein. However, as to the nonlinear Langevin equation involving two fractional orders, the research work is still in its infancy and is focused on boundary value problems. The Dirichlet boundary value problem was studied in [13] with the limitation that $0 < \alpha, \beta \leq 1$, while the three-point boundary value problem was studied in [14] under the condition that $0 < \alpha \leq 1, 1 < \beta \leq 2$. The study of the corresponding initial value problems has not been reported.

In this paper, we discuss the existence and uniqueness of solutions for the following initial value problem of Langevin equation involving two fractional orders:

$$\begin{cases} ^{c}{}_{0}\mathcal{D}_t^{\beta}(^{c}{}_{0}\mathcal{D}_t^{\alpha} + \gamma)x(t) = f(t, x(t)), & 0 < t < 1, \\ x^k(0) = \mu_k, & 0 \leqslant k < l, \\ x^{\alpha+k}(0) = \nu_k, & 0 \leqslant k < n, \end{cases} \tag{1.1}$$

where $^{c}{}_{0}\mathcal{D}_t^{\alpha}$ and $^{c}{}_{0}\mathcal{D}_t^{\beta}$ are the Caputo fractional derivatives, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function and $\gamma \in \mathbb{R}, m, n \in \mathbb{N}^+, m-1 < \alpha \leqslant m, n-1 < \beta \leqslant n, l = \max\{m, n\}$. We give no limitation to the fractional orders α and β, so the results can be applied to a wide space.

The rest of the paper is organized as follows. In Section 2, we provide the methods employed in obtaining the existence results and the auxiliaries on fractional differentiation and integration, limited to the Caputo fractional derivative. In Section 3, we prove the main results. The existence results follow from the Leray–Schauder Nonlinear Alternative and the uniqueness of solutions is a consequence of the Banach contraction principle. We give conclusion in Section 4.

2. Preliminaries

In this section, we present some notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Definition 2.1 [15]. Let us assume that $x(t) \in C[a, b], p \in \mathbb{R}^+$, then the Riemann–Liouville integral of order p is the expression

$$_{a}\mathcal{I}_t^{p}x(t) = \int_a^t \frac{(t-u)^{p-1}}{\Gamma(p)} x(u)\mathrm{d}u, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler's gamma function, the subscripts a and t denote the two limits related to the operation of fractional integral.

Definition 2.2 [15]. Let us assume that $x(t) \in C^n[a, b], n \in \mathbb{N}^+, p \in (n-1, n)$, then the *Caputo* fractional derivative of order p is the expression

$$^{c}{}_{a}\mathcal{D}_t^{p}x(t) = \int_a^t \frac{(t-u)^{n-p-1}}{\Gamma(n-p)} x^n(u)\mathrm{d}u, \tag{2.2}$$

We remark that the Caputo derivative becomes the conventional nth derivative of the function as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer-order derivatives.

The relationship between (2.1) and (2.2) and some of their properties are stated in the following lemmas.

Lemma 2.1 [15]. *Let $p, q \in \mathbb{R}^+, r, l \in \mathbb{R}, m, n \in \mathbb{N}^+, p \in (n-1, n), u \in C[a, b], f \in C^{m+n}[a, b]$. Then,*

(1) $_{a}\mathcal{I}_t^{0} := I, \quad {_{a}\mathcal{I}_t^{p}}\,{_{a}\mathcal{I}_t^{q}}u(t) = {_{a}\mathcal{I}_t^{q}}\,{_{a}\mathcal{I}_t^{p}}u(t) = {_{a}\mathcal{I}_t^{p+q}}u(t);$

(2) $^{c}{}_{a}\mathcal{D}_t^{0} := I, \quad ^{c}{}_{a}\mathcal{D}_t^{p}C = 0, \quad ^{c}{}_{a}\mathcal{D}_t^{p}\,{_{a}\mathcal{I}_t^{p}}u(t) = u(t);$

(3) $_{a}\mathcal{I}_t^{p}(t-a)^r = \frac{\Gamma(r+1)}{\Gamma(r+1+p)}(t-a)^{r+p}, \quad ^{c}{}_{a}\mathcal{D}_t^{p}(t-a)^l = \frac{\Gamma(l+1)}{\Gamma(l+1-p)}(t-a)^{l-p};$