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Lie group symmetries and Riemann function of Klein–Gordon–Fock equation with central symmetry

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ABSTRACT

In the present paper Lie symmetry group method is applied to find new exact invariant solutions for Klein–Gordon–Fock equation with central symmetry. The found invariant solutions are important for testing finite-difference computational schemes of various boundary value problems of Klein–Gordon–Fock equation with central symmetry. The classical admitted symmetries of the equation are found. The infinitesimal symmetries of the equation are used to find the Riemann function constructively.

1. Introduction

The Klein–Gordon–Fock (KGF) equation with central symmetry has form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r} - \frac{b}{r^2} w(r,t), \tag{1}$$

where b is a real parameter. Making the point change of variables $w(r,t) = u(r,t)/r$ (the independent variables do not change) we reduce Eq. (1) to more simple form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} - \frac{b}{r^2} u(r,t). \tag{2}$$

Eq. (1) as well as (2) are commonly encountered in different classical and quantum physical problems with central symmetry. For example, Eq. (2) appears in electromagnetics to describe a time evolution of transient electromagnetic fields in homogeneous media and biconical transmission lines [1–3]. Therefore, both Eqs. (1) and (2) have been studied repeatedly.

All inequivalent coordinate systems providing separation of variables in Eq. (2) have been found in [4–6]. In these works Eq. (2) is classified as wave equation with special time-independent potential.

Eqs. 1,2 were investigated more completely in the special case when the parameter is $b = n(n+1)$, where $n = 1, 2, 3, \ldots$. In such case the general solution of Eq. (1) is known [7]

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$$u(r,t) = r^n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left[\frac{\Psi(r-t)+\Phi(r+t)}{r}\right], \tag{3}$$

where $\Psi(\cdot)$ and $\Phi(\cdot)$ are arbitrary sufficiently smooth functions. Also the integral (Laplace, Weber) transform method can be successfully applied to solve in closed form some boundary value problems for Eq. (1) or (2) only if $b = n(n+1)$. This follows from the fact that in this case inverse integral transformations contain the kernels expressed via modified Bessel functions with half-integer index which can be transformed into exponential functions. In the more general case when the parameter $b \neq n(n+1)$ is some arbitrary real value the general solution (3) contains fractional derivative and the inverse integral transformations contain the complicated kernels. The peculiarity of this special case can be explained as follows. On the family of equations parameterized by the constant acts Bäcklund transform that changes the parameter. Therefore, in this family, there is a countable subset of equations with the parameter $b = n(n+1)$ that can be reduced to the d'Alembert equation $(b = 0)$ and a general solution can be found in closed form (3).

From the physical point of view, values of the parameter $b = n(n+1)$ correspond to electromagnetic fields in the free space. In the case of a conical metal line the parameter b can be equal to some arbitrary positive value. Thus, closed form analytical solution of Eq. (2) can be found for a narrow class of physical problems with the parameter $b = n(n+1)$. In fact interesting boundary value problems for Eq. (2) are solved numerically by means of the finite-difference time domain (FDTD) method [3]. But applying FDTD to solve equations (1), (2) one should be careful near the singular point $r = 0$ where numerical computational scheme can give unstable results. Therefore, there is a great need for the exact reference solutions of Eqs. 1,2 with the arbitrary parameter b to test the FDTD computational schemes.

The main purpose of this paper is to find new suitable reference solutions of Eq. (2) using the Lie symmetry group (group analysis) method. The used method of group analysis of differential equations is based on the Lie–Ovsiannikov approach [8–11]. In the second section making standard technical calculations we obtain infinitesimal operators and local one-parameter groups of point transformations for Eq. (2). In the third section the well-known classification of the non-equivalent one-dimensional subalgebras for four-dimensional algebra of Eq. (2) is used to find novel invariant solutions of Eq. (2) which are convenient for testing and verification of numerical computational schemes of Eq. (2). In the fourth section we show how the Riemann function of (2) can simply be computed via infinitesimal symmetries.

2. Symmetries of KGF equation with central symmetry

The infinitesimal operator of the local Lie group of point transformations which are admitted by Eq. (2) is

$$X = \xi^1(r,t,u)\frac{\partial}{\partial t} + \xi^2(r,t,u)\frac{\partial}{\partial r} + \eta(r,t,u)\frac{\partial}{\partial u}. \tag{4}$$

The second prolongation of the operator (4) has form

$$\underset{2}{X} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_r} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{12}\frac{\partial}{\partial u_{tr}} + \zeta_{22}\frac{\partial}{\partial u_{rr}}. \tag{5}$$

Applying Lie Theorem about an invariance condition we obtain

$$\underset{2}{X}\left(u_{tt} - u_{rr} + \frac{bu}{r^2}\right)\Big|_{u_{tt}=u_{rr}-\frac{bu}{r^2}} = \zeta_{11} - \zeta_{22} - \frac{2bu}{r^3}\xi^2 + \frac{b}{r^2}\eta\Big|_{u_{tt}=u_{rr}-\frac{bu}{r^2}} = 0. \tag{6}$$

Using the well-known prolongation formula [8–11] one can express the functions ζ_{11} and ζ_{22} via the components of the vector field ξ^1, ξ^2 and η. For example, we have

$$\zeta_{11} = \eta_{tt} + (2\eta_{ut} - \xi^1_{tt})u_t - \xi^2_{tt}u_r + (\eta_{uu} - 2\xi^1_{tu})u_t^2 - 2\xi^2_{ut}u_t u_r - \xi^1_{uu}u_t^3 - \xi^2_{uu}u_t^2 u_r - 2u_{tr}\left[\xi^2_t + \xi^2_u u_t\right]$$
$$+ u_{tt}\left[\eta_u - 2\xi^1_t - \xi^2_u u_r - 3\xi^1_u u_t\right].$$

Substituting the functions ζ_{11} and ζ_{22} in (6) we obtain the determining equations to find the symmetry of Eq. (2). Derivation of the determining equations for various differential equations is described in detail in books [8–11]. Solving the obtained determining equations for Eq. (2) we find the vector field

$$\xi^1(r,t) = \frac{C_1}{2}(t^2 + r^2) + C_2 t + C_3, \quad \xi^2(r,t) = (C_1 t + C_2)r, \quad \eta(r,t,u) = C_4 u + \beta(r,t). \tag{7}$$

Here $C_1, \ldots, C_4$ are arbitrary constants and $\beta(r,t)$ is arbitrary solution of Eq. (2). Choosing the appropriate constants $C_1, \ldots, C_4$ we have the following

Proposition. The maximal invariance Lie algebra of Eq. (2) is an infinite-dimensional algebra which contains a four-dimensional nontrivial subalgebra. The algebra basis elements are presented by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad X_3 = (t^2 + r^2)\frac{\partial}{\partial t} + 2rt\frac{\partial}{\partial r}, \quad X_4 = u\frac{\partial}{\partial u}, \quad X_\beta = \beta(r,t)\frac{\partial}{\partial u}. \tag{8}$$