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Conservation laws of inviscid Burgers equation with nonlinear damping

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ABSTRACT

In this paper, the new conservation theorem presented in Ibragimov (2007) [14] is used to find conservation laws of the inviscid Burgers equation with nonlinear damping $u_t + g(u)u_x + \lambda h(u) = 0$. We show that this equation is both quasi self-adjoint and self-adjoint, and use these concepts to simplify conserved quantities for various choices of $g(u)$ and $h(u)$.

1. Introduction

Conservation laws play vital roles in obtaining in-depth understanding of physical properties of various systems. They often aid to the choice of solution methods, and are keys to obtaining solutions of simple, poorly understood or complex systems. Due to the unparallel importance of conserved quantities to various fields of science, many methods that can be used to compute them have been developed by various researchers. For partial differential equations (PDEs) with variational principles, Noether [22] developed a systematic method of finding conservation laws. But the restriction to Euler–Lagrange equations prevent Noether's theorem to be applied to evolution equations, to differential equations of odd order, etc. This predicament led to the development of various generalizations of Neother's method which enable the conservation laws of non variational PDEs to be found, see [2,3,11,14,19–21].

In [13,14], a new conservation theorem that is based on the concept of self-adjoint equations was proposed. It enables the conservation laws of any arbitrary differential equation without a lagrangian to be found, see [6,18]. Since not all differential equations are self-adjoint, the concept in [14] has been extended to quasi self-adjointness [15,16,24], nonlinear self-adjointness [17], and weak self-adjointness [9,10].

In this paper, we shall use the method proposed by Ibragimov [14] to find conservation laws for the damped inviscid Burgers equation

$$u_t + g(u)u_x + \lambda h(u) = 0, \tag{1}$$

where $\lambda h(u)$ is the nonlinear damping term. Both $g(u)$ and $h(u)$ are assumed to be smooth functions of $u = u(x,t)$ with non vanishing first derivatives and $\lambda \neq 0$ is a constant. The case for $\lambda = 0$ has been carried out in [1,7,8].

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The outline of the paper is as follows. We review some definitions and theorem in the next section, and also establish conditions for quasi self-adjointness and self-adjointness. In Section 3, we find nontrivial conserved quantities of Eq. (1) for various choices of $g(u)$ and $h(u)$. Section 4 deals with the concluding remarks.

2. Quasi self-adjointness and self-adjointness conditions

In order to establish quasi self-adjointness and self-adjointness conditions that will enable the simplification of computed conserved quantities, it will be useful to recall the following definitions and results as presented in [13–15]. Since we shall be dealing with first-order Lagrangian, only those relevant to our case will be presented.

Definition 2.1. The Euler–Lagrange operator (variational derivatives), $\frac{\delta}{\delta u}$, is defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - \cdots, \tag{2}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_{ij}} + \cdots \tag{3}$$

is the total differentiation with respect to x^i.

Definition 2.2. Consider a first-order PDE

$$F(x, u, u_1) = 0, \tag{4}$$

where $F(x, u, u_1) \in \mathcal{A}$ are differential functions with n independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable $u = u(x)$. Let

$$F^*(x, u, v, u_1, v_1) = \frac{\delta(vF)}{\delta u} \tag{5}$$

be a differential function, where $v = v(x)$ is the new dependent variable. The adjoint equation to (4) is defined by

$$F^*(x, u, v, u_1, v_1) = 0 \tag{6}$$

Definition 2.3. The first-order PDE (4) is said to be self-adjoint if

$$F^*|_{v=u} = 0$$

is identical to the original Eq. (4). That is,

$$F^*(x, u, u, u_1, u_1) = \rho(x, u, \ldots) F(x, u, u_1). \tag{7}$$

Definition 2.4. The first-order PDE (4) is said to be quasi self-adjoint if

$$F^*|_{v=\zeta(u)} = \rho(x, u, \ldots) F(x, u, u_1) \tag{8}$$

for some arbitrary function $\zeta(u)$ with $\zeta'(u) \neq 0$.

Theorem 2.1. *The first-order PDE (4) considered together with its adjoint equation (6) has a Lagrangian given by*

$$\mathcal{L} = vF(x, u, u_1). \tag{9}$$

Theorem 2.2. *Consider an equation*

$$F(x, u, u_1) = 0 \tag{10}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u. The adjoint equation

$$F^*(x, u, v, u_1, v_1) = \frac{\delta(vF)}{\delta u} = 0 \tag{11}$$

to Eq. (10) inherits the symmetries of Eq. (10). Namely, if Eq. (10) admits an operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}, \tag{12}$$