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Traveling waves, impulses and diffusion chaos in excitable media

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ABSTRACT

In the present work it is shown, that the FitzHugh–Nagumo type system of partial differential equations with fixed parameters can have an infinite number of different stable wave solutions, traveling along the space axis with arbitrary speeds, and also traveling impulses and an infinite number of different states of spatiotemporal (diffusion) chaos. Those solutions are generated by cascades of bifurcations of cycles and singular attractors according to the FSM theory (Feigenbaum–Sharkovskii–Magnitskii) in the three-dimensional system of ordinary differential equations (ODEs), to which the FitzHugh–Nagumo type system of equations with self-similar change of variables can be reduced.

1. Introduction

A large class of physical, chemical and biological media much studied by the methods of nonlinear and chaotic dynamics is described by the reaction–diffusion system of partial differential equations:

$$u_t = D_1 u_{xx} + f(u, v, \mu), \quad v_t = D_2 v_{xx} + g(u, v, \mu) \tag{1}$$

dependent on a scalar parameter μ. As a rule, there is a positive feedback on one of the variables in the systems of the form (1). Such a variable is called an activator. The second variable which slows down the increase (development) of the activator is called an inhibitor.

The FitzHugh–Nagumo type systems of differential equations describing nonlinear processes occurring in so-called excitable media are a special case of reaction–diffusion systems. Those are the propagation of impulses in nerve membrane and cardiac muscle [1–4] and different types of autocatalytic chemical reactions [5–6]. Slow diffusion of one variable in the reaction–diffusion system as compared with another variable is the basic property describing the class of excitable media. Therefore, the FitzHugh–Nagumo type system can be written in the following form:

$$u_t = Du_{xx} + f(u, v, \mu), \quad v_t = g(u, v, \mu), \tag{2}$$

where μ is a parameter vector in general case. It is well known that systems of the form (2) can exhibit switching waves, traveling waves and traveling impulses and dissipative spatially inhomogeneous stationary structures, and also irregular nonperiodic nonstationary structures, sometimes called biological (or chemical) turbulence.

The analysis of regular solutions of (2) in one-dimensional case can be carried out using a self-similar change of variables $\xi = x - ct$ and transition to the three-dimensional system of ODEs:

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$$\dot{u} = y, \quad \dot{y} = -(cy + f(u, v, \mu))/D, \quad \dot{v} = -g(u, v, \mu)/c \tag{3}$$

where derivatives are taken with respect to the variable ξ. The switching wave in the system (2) is described by the separatrix of the system (3), moving from one singular point of the system into another singular point. Traveling wave and traveling impulse of the system (2) are described by the limit cycles and the separatrix loop of the singular point of the system (3).

This work shows that the diffusion chaos (turbulence) in the FitzHugh–Nagumo type system (2) is described by singular attractors of the ODE system (3) according to the FSM theory [7–10].

For this purpose as follows from the FSM theory the dissipative ODE system (3) with nonlinear functions $f(u, v, \mu)$ and $g(u, v, \mu)$ should have a saddle-focus type singular point, and also subharmonic and homoclinic cascades of bifurcations of singular cycles, converging to a homoclinic separatrix loop of that singular point.

2. A model of an excitable medium

Let us consider a special case of systems of Eqs. (2) and (3) with the following nonlinearities:

$$f(u, v, \mu) = -(u-1)(u-\delta v)/\varepsilon, \quad g(u, v, \mu) = arctg\,(\alpha u) - v, \tag{4}$$

where ε is a small parameter. Notice that the system of Eq. (4) with a polynomial in the two variables (u, v) function $f(u, v, \mu)$ and with the function $g(u, v, \mu)$ with finite limit values for every v as $u \to \pm\infty$ describes certain types of autocatalytic chemical reactions [6]. It is easy to see that the system (3) and (4) has the singular point $O(0, 0, 0)$ for any parameter values. In addition, the system (3) and (4) has two symmetric singular points $O_{\pm}(\pm u_*, 0, \pm u_*/\delta)$ for $\alpha > 1/\delta$, where u_* is a positive solution to the equation $\delta\, arctg\,(\alpha u_*) = u_*$. Therefore, the system (3) and (4) for different values of parameters $(D, \alpha, \delta, \varepsilon, c)$ can have subharmonic, homoclinic and heteroclinic cascades of bifurcations of stable cycles, converging to homoclinic and heteroclinic separatrix contours of singular points. Every such subharmonic, homoclinic or heteroclinic cascade of bifurcations contains an infinite number of different singular attractors of the ODE system (3) and accordingly an infinite number of nonperiodical nonstationary spatially inhomogeneous (chaotic, turbulent) states of behavior for the solutions of the original FitzHugh–Nagumo type system (2).

The case in which the implicit system parameter c, the propagation velocity of perturbations along the x axis, is the bifurcation parameter is of great interest. That case implies that the FitzHugh–Nagumo type system (2) with fixed parameters can have an infinite number of different autowave solutions of arbitrary periods, traveling along the space axis with different velocities, and also can have an infinite number of different states of spatiotemporal (diffusion) chaos.

Let us illustrate the last statement by an example of the system (3) and (4) with fixed parameter values $D = 1, \alpha = 2, \delta = 6, \varepsilon = 0.195$. For those parameter values zero singular point O of the system (3) and (4) is a stable focus for $c > \sqrt{1 + (\alpha\delta - 1)/(1-\varepsilon)} \approx 3.83$. For lower parameter c the limit cycle is originated as a result of the Andronov–Hopf bifurcation, and it stays stable up to $c \approx 2.9635$. The singular point O itself becomes a saddle-focus. With a decrease in parameter c the Feigenbaum cascade of period doubling bifurcations of stable limit cycles is implemented in the system (3) and (4) up to the formation of the first singular attractor – the Feigenbaum attractor for $c \approx 2.874$.

With further decrease in parameter c the complete subharmonic cascade of bifurcations of stable cycles according to the Sharkovskii's order is implemented in the system (3) and (4) and then the incomplete homoclinic cascade of bifurcations of stable cycles is implemented, converging to a homoclinic contour – the separatrix loop of the saddle-focus O in the system (3) and (4) (Fig. 1).

However, the saddle-focus separatrix loop itself cannot be revealed with given parameters in the system (3) and (4). For revealing of the surface of existence of the saddle-focus separatrix loop in many-dimensional space of the parameters of the system (3) and (4) one can use the method, developed by one of the present paper's authors and described in works [7], where examples of revealing of the surfaces and curves of existence of homoclinic and heteroclinic singular point contours in the Lorenz system, including the homoclinic and heteroclinic butterflies, are considered.

Fig. 2 illustrates waves traveling along the space axis x of the system of equations of the excitable medium (2), and (4), corresponding with the homoclinic cycles of period 3 and 5 of the self-similar ODE system (3) and (4). In Fig. 2 one period of the solution $u(x - ct)$ of the system (2), and (4) in some fixed moment of time $t = 0$ is presented. The traveling impulse in the excitable medium is apparently generated by the homoclinic separatrix loop of the saddle-focus type ODE system singular point.

3. System describing the oxidation of CO on a Pt(110) surface

An example of a real excitable medium is the catalytic CO oxidation on the Pt(110) surface, for which experiments have revealed a rich variety of spatiotemporal structures on the catalyst surface, such as traveling impulses, spiral waves and chemical turbulence (diffusion chaos). The chemical kinetics of CO oxidation on Pt (110) surface is described by three equations: the balance equation for the concentration of O on the Pt surface, a similar equation for the concentration of CO and an equation for the change of state of the Pt surface, which is coverage dependent and modeled by a function $f(u)$ (see below). After adding to those equations the CO diffusion and adiabatic elimination of the variable representing the oxygen coverage, the following special case of two-component FitzHugh–Nagumo type reaction–diffusion equations are obtained [6]: