Commun Nonlinear Sci Numer Simulat 19 (2014) 1836–1858

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Building fast well-balanced two-stage numerical schemes for a model of two-phase flows

Mai Duc Thanh *

Department of Mathematics, International University, Quarter 6, Linh Trung Ward, Thu Duc District, Ho Chi Minh City, Viet Nam

ARTICLE INFO

Article history:
Received 23 April 2013
Received in revised form 15 October 2013
Accepted 17 October 2013
Available online 30 October 2013

Keywords:
Two-phase flow
Well-balanced scheme
Lax–Friedrichs scheme
Richtmyer's scheme
Roe scheme

ABSTRACT

We present a set of well-balanced two-stage schemes for an isentropic model of two-phase flows arisen from the modeling of deflagration-to-detonation transition in granular materials. The first stage is to absorb the source term in nonconservative form into equilibria. Then in the second stage, these equilibria will be composed into a numerical flux formed by using a convex combination of the numerical flux of a stable Lax–Friedrichs-type scheme and the one of a higher-order Richtmyer-type scheme. Numerical schemes constructed in such a way are expected to get the interesting property: they are fast and stable. Tests show that the method works out until the parameter takes on the value *CFL*, and so any value of the parameter between zero and this value is expected to work as well. All the schemes in this family are shown to capture stationary waves and preserves the positivity of the volume fractions. The special values of the parameter $0, 1/2, 1/(1+CFL)$, and *CFL* in this family define the Lax–Friedrichs-type, FAST1, FAST2, and FAST3 schemes, respectively. These schemes are shown to give a desirable accuracy. The errors and the CPU time of these schemes and the Roe-type scheme are calculated and compared. The constructed schemes are shown to be well-balanced and faster than the Roe-type scheme.

1. Introduction

In this paper we aim to build a set of fast and well-balanced schemes for the following isentropic model of two-phase flows,

$$\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) = -p_g\partial_x\alpha_g,\\
&\partial_t\rho_s + \partial_x(\rho_s u_s) = 0, \quad x \in R, t > 0.
\end{aligned} \tag{1.1}$$

The model (1.1) is obtained from the two-phase mixture model to study the deflation-to-detonation transition in granular explosives, see [7], by simplifying the model. An alternative form of the model was presented in [4], in which the compaction dynamics equation (the last equation of (1.1)) has a different form:

$$\partial_t\alpha_g + u_s\partial_x\alpha_g = 0,$$

Observe that the last equation involves a *nonconservative term* $u_s\partial_x\alpha_g$. The reader is referred to [10] for the mathematical formulation of nonconservative hyperbolic systems involving nonconservative terms.

* Tel.: +84 8 2211 6965; fax: +84 8 3724 4271.
E-mail address: mdthanh@hcmiu.edu.vn

1007-5704/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cnsns.2013.10.017

Throughout, we use the subscripts g and s to indicate the quantities in the g-phase (referred to as the gas phase), and in the solid phase (referred to as the solid phase), respectively. However, our study in this work can be applied for more general materials. For example, one of the two phases or both may be liquid. The notations $\alpha_k,\ \rho_k,\ u_k,\ p_k,\ k = g, s$, stand for the volume fraction, density, velocity, and pressure in the k-phase, $k = g, s$, respectively. The volume fractions are constraint by the relation

$$\alpha_s + \alpha_g = 1. \tag{1.2}$$

In our recent work [40], a Roe-type scheme was constructed by using the admissible solid contacts at each node to absorb the nonconservative terms. The states on both sides of these solid contacts are incorporated into a Roe-type matrix of the decoupling system of (1.1) which is obtained from (1.1) by letting the volume fractions be constant. A similar process could also be made by using another numerical flux, for example the one of the Lax–Friedrichs scheme to form a Lax–Friedrichs-type scheme. However, this Lax–Friedrichs-type scheme has less accuracy then the Roe-type scheme, as seen in Section 4. This is probably because the Lax–Friedrichs scheme is, though stable, too diffusive. Furthermore, the incorporation of the states on both sides of the solid contacts mentioned above into a higher-order scheme such as the Lax–Wendroff or Richmyer's scheme does not yield satisfactory results: the scheme is numerically unstable, where large oscillations appear shortly in numerical tests. Customarily, a well-balanced scheme is the one which can preserve the steady state solution exactly.

Motivated by the above argument, we aim to build in this paper a set of numerically stable schemes that can be faster and have a better accuracy than the Roe-type scheme. For this purpose, we form a one-parameter family of numerical fluxes by using convex combinations of the numerical fluxes of the Lax–Friedrichs scheme and the second-order Richtmyer's with a parameter $\theta \in [0, 1]$. The states $U^n_{j\pm1,\mp}$ on the other side of the solid contacts at $x_{j\pm1/2}$ from any given states $U^n_{j\pm1}$ are then incorporated in numerical fluxes in this family to produce new schemes. The values $\theta = 0, 1/2, \theta = 1/(1 + CFL)$ and CFL in this family define a Lax–Friedrichs-type, FAST1, FAST2, and FAST3 schemes, respectively. Thus, the FAST1 and FAST2 schemes are formed in a similar way as the FORCE and GFORCE schemes. Recall that the FORCE and GFORCE schemes are convex combinations of the Lax–Friedrichs scheme and Lax–Wendroff scheme, see [29,30,9]. Tests of Lax–Friedrichs-type, FAST1, FAST2, and FAST3 schemes are presented. Errors, order of convergence, numbers of iterations, CPU time are evaluated.All the tests show desirable approximations to the exact solutions. The results are compared with a newly constructed Roe-type scheme [40]. Tests show that the FAST3 scheme gives the better results than the Roe-type scheme. Naturally, the same result is expected for schemes corresponding to the values of the parameter θ closed to CFL, at least. Moreover, we show that our schemes are well-balanced. Observe that the restrictive attention to the isentropic case may cause a certain limitation for applications, and would motivate for future developments for the general case.

We note that numerical approximations of nonconservative systems have attracted attention of many authors. Numerical well-balanced schemes for a single conservation law with a source term are presented in [15,16,5,6,3]. Numerical schemes for multi-phase multi-pressure models were presented in [21,25,1,26,13,39,36]. Various numerical schemes for two-fluid models of two-phase flows were constructed in [42,38,35]. Well-balanced schemes for other nonconservative hyperbolic systems were built in [20,19,24,3,37,8,17].The Riemann problem for various nonconservative hyperbolic systems was studied in [22,33,23,14,32,2]. Shock waves in two-fluid models of two-phase flows were studied in [18,34]. Some recent Godunov-type schemes for various fluid flow models are presented in [31,27,24,28]. See also the references therein.

The organization of this paper is as follows. In Section 2 we present basic concepts of the system (1.1): non-strict hyperbolicity, discontinuities, and admissible solid contact waves. In Section 3 we construct a one-parameter family of well-balanced schemes. First, we describe a family of numerical fluxes by using convex combinations of the numerical fluxes of a stable Lax–Friedrichs and a higher-order Richmyer's schemes. Then, we show how to incorporate admissible solid contacts into these numerical fluxes to obtain a numerical scheme. Section 4 is devoted to numerical tests, where tests for the well-balanced Lax–Friedrichs-type scheme, FAST1, FAST2, FAST3, and Roe-type schemes are carried out. Finally, in Section 5 we draw several conclusions and discussions.

2. Preliminaries

2.1. Nonstrict hyperbolicity

The governing Eqs. (1.1) may be re-written as a system in nonconservative form:

$$U_t + A(U)U_x = 0, \tag{2.1}$$

where

$$U = \begin{pmatrix} \rho_g \\ u_g \\ \rho_s \\ u_s \\ \alpha_g \end{pmatrix}, \qquad A(U) = \begin{pmatrix} u_g & \rho_g & 0 & 0 & \frac{\rho_g(u_g-u_s)}{\alpha_g} \\ h'_g(\rho_g) & u_g & 0 & 0 & 0 \\ 0 & 0 & u_s & \rho_s & 0 \\ 0 & 0 & h'_s(\rho_s) & u_s & \frac{p_g-p_s}{\alpha_s\rho_s} \\ 0 & 0 & 0 & 0 & u_s \end{pmatrix} \tag{2.2}$$