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Nonlinear force density method for the form-finding of minimal surface membrane structures

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ABSTRACT

We develop an alternative approach for the form-finding of the minimal surface membranes (including cable membranes) using discrete models and nonlinear force density method. Two directed weighted graphs with 3 and 4-sided regional cycles, corresponding to triangular and quadrilateral finite element meshes are introduced as computational models for the form-finding problem. The triangular graph model is closely related to the triangular computational models available in the literature whilst the quadrilateral graph uses a novel averaging approach for the form-finding of membrane structures within the context of nonlinear force density method. The viability of the mentioned discrete models for form-finding are studied through two solution methods including a fixed-point iteration method and the Newton–Raphson method with backtracking. We suggest a hybrid version of these methods as an effective solution strategy. Examples of the formation of certain well-known minimal surfaces are presented whilst the results obtained are compared and contrasted with analytical solutions in order to verify the accuracy and viability of the suggested methods.

1. Introduction

Tension membrane structures are efficient, attractive and aesthetic structural forms which cover large areas including exhibition centres, auditoriums, stadiums, etc. Minimal surfaces, physically realised by soap-films, may be considered as viable initial forms for tension membrane structures. One of the major steps in the analysis of tension membrane structures is the form-finding. Form-finding refers to a process which provides us with a self-equilibrated state for a pre-stressed structure. For tension membrane structures reinforced by cables (cable–membranes), the equilibrium between prescribed force of boundary cables, prescribed membrane stresses and forces induced at the fixed supports leads to a structural form. Over the decades, different methods have been proposed for the form-finding problems either for cable-net structures and tension membranes. Recently, Veenendaal and Block [1] presented a comprehensive review of the form-finding methods. The earlier study suggests that all the form-finding methods may be more or less categorised into three groups, namely the stiffness matrix methods, geometric stiffness methods and dynamic equilibrium methods. However, since the methods proposed in this paper fall into the second category (i.e., geometric stiffness methods), we briefly review the main contributions and methods in this category as follows.

The geometric stiffness methods, as the name suggests, are form-finding methods which employ geometric stiffness matrices which are independent of the material properties. This category undoubtedly includes the most appealing form-finding methods. The earliest method from this category was developed by Linkwitz and Schek [2] and Schek [3] and is

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known as the force density method (FDM). The original FDM has been designed for cable-net structures. Since 1971, several form-finding methods have been developed for tension membrane structures which may be classified in this category, including the assumed geometric stiffness method of Haber and Abel [4], the surface stress density method (SSDM) of Maurin and Motro [5,6]), the updated reference strategy (URS) of Bletzinger and Ramm [7] (see also Bletzinger et al. [8] and Wüchner and Bletzinger [9]), the revised geometric stiffness method of Nouri-Baranger [10,11], the preliminary form-finding surface fitting (PFSM) approach of Sánchez et al. [12] and the natural force density method (NFDM) of Pauletti and Pimenta [13].

In this paper, first, we describe the well-known FDM for the form-finding of cable nets. The extension of the method to the form-finding of membranes is then presented using a conventional triangular finite element. Although the mentioned formulation is well documented in the literature, we present a rather different graph theoretical representation of this formulation. In other words, we simulate a tension membrane structure using a weighted directed graph (graph with 3 sided regional cycles corresponding to the triangular finite element meshes) where each edge of the graph is assigned an explicit weight (sum of force densities acting on that edge), simply calculated from current geometry. Another discrete model, the graph model of a quadrilateral finite element mesh, is then introduced based on the triangular graph model and using a novel averaging approach. These graphs are then used as discrete computational models for the form-finding of minimal surface membranes. The method offered promises with regards to forming minimal surface membranes in a very similar way to the form-finding of cable-net structures. Second, two solution methods including a fixed-point iteration method and a Newton Raphson method with backtracking are suggested for the form-finding through our formulations – a hybrid and effective version of these methods is also proposed. Finally, examples including formation of a *catenoid*, *Scherk* and *Schwarz P* surfaces are presented using our methods, whilst the results obtained are compared and contrasted with those analytical solutions available in the literature. The comparisons clearly verify the accuracy and viability of our approach.

2. FDM for cable nets

The FDM was first proposed by Linkwitz and Schek [2] (see also [3]) and has been widely used in the form-finding of cable nets and tensegrity structures. Considering a prescribed value for the force density of cables (q_i), the form-finding of a cable-net with n nodes and m elements is simply formulated as the solution of a system of linear equations as follows

$$\mathbf{BQB}^t\mathbf{X} = \mathbf{P} \tag{1}$$

in which $\mathbf{B} = [b_{ij}]$ is defined as

$$b_{ij} = \begin{cases} -1 & \text{if} \quad i \text{ is the start node of element } j \\ 1 & \text{if} \quad i \text{ is the end node of element } j \\ 0 & \quad \text{Otherwise} \end{cases} \tag{2}$$

From graph theoretical point of view, $\mathbf{B}$ is called the node-element incidence matrix of a directed graph where each element is directed from node i to $j(i < j)$. Furthermore, $\mathbf{X} = [\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is the matrix of nodal coordinates, whilst $\mathbf{P} = [\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z]$ is the matrix of external nodal forces and $\mathbf{Q} = diag(\mathbf{q})$, $\mathbf{q} = [q_1, q_2, \ldots, q_m]^t$ is a diagonal matrix consisting of the force density of elements. In Eq. (1), $\mathbf{BQB}^t$ is also known as a weighted Laplacian matrix where the cable-net is represented as a directed weighted simple graph. In the process of form-finding we usually have certain fixed nodes (support conditions) as well as external forces which are considered to be zero. Therefore, Eq. (1) can be slightly changed to deal with these cases. Let the nodal coordinate matrix be partitioned into $\mathbf{X}_f$ (nodal coordinates of free nodes) and $\mathbf{X}_r$ (nodal coordinates of restrained nodes). As a result, $\mathbf{B}$ and $\mathbf{P}$ in Eq. (1) are partitioned accordingly as shown in Eq. (3).

$$\begin{bmatrix} \mathbf{B}_f \\ \mathbf{B}_r \end{bmatrix} \mathbf{Q} \begin{bmatrix} \mathbf{B}_f^t & \mathbf{B}_r^t \end{bmatrix} \begin{bmatrix} \mathbf{X}_f \\ \mathbf{X}_r \end{bmatrix} = \begin{bmatrix} \mathbf{P}_f \\ \mathbf{P}_r \end{bmatrix} \tag{3}$$

The equilibrium equations can then be written based on nodal coordinates of free nodes as follows.

$$(\mathbf{B}_f\mathbf{QB}_f^t)\mathbf{X}_f = \mathbf{P}_f - (\mathbf{B}_f\mathbf{QB}_r^t)\mathbf{X}_r \tag{4}$$

With regards to form-finding, we are usually interested in a self-equilibrium state ($\mathbf{P}_f = \mathbf{0}$) and thus the final system of linear equations for the form-finding of cable nets is obtained as

$$\mathbf{GX}_f = \mathbf{F}_f \tag{5}$$

where,

$$\mathbf{G} = \mathbf{B}_f\mathbf{QB}_f^t, \quad \mathbf{F}_f = -(\mathbf{B}_f\mathbf{QB}_r^t)\mathbf{X}_r \tag{6}$$