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Coefficient characterization of linear differential equations with maximal symmetries

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ABSTRACT

A characterization of the general linear equation in standard form admitting a maximal symmetry algebra is obtained in terms of a simple set of conditions relating the coefficients of the equation. As a consequence, it is shown that in its general form such an equation can be expressed in terms of only two arbitrary functions, and its connection with the Laguerre–Forsyth form is clarified. The characterizing conditions are also used to derive an infinite family of semi-invariants, each corresponding to an arbitrary order of the linear equation. Finally a simplifying ansatz is established, which allows an easier determination of the infinitesimal generators of the induced pseudo group of equivalence transformations, for all the three most common canonical forms of the equation.

1. Introduction

By a result of Lie [1], a linear ordinary differential equation (ODE) of a general order n is known to have a symmetry algebra of maximal dimension d_n if it is reducible by a point transformation to the equation $y^{(n)} = 0$, which will henceforth be referred to as the canonical form of the linear equation. In a much recent paper Krause and Michel [2] proved the converse of this result and also showed that a linear equation is iterative if and only if its symmetry algebra has the maximal dimension d_n. (By the cited result of Lie [1], $d_n = n + 4$ for $n \geqslant 3$). Characterizing linear equations having a symmetry algebra of maximal dimension is therefore the same as characterizing linear equations that are reducible by a point transformation to the canonical form. The latter characterization for the third-order equation $y^{(3)} + c_2 y'' + c_1 y' + c_0 y = 0$ is due to Lie [3] and Laguerre [4] who showed independently that this equation is reducible to the canonical form if and only if its coefficients satisfy the equation

$$54c_0 - 18c_1c_2 + 4c_2^3 - 27c_1' + 18c_2c_2' + 9c_2'' = 0. \tag{1}$$

This characterization also clearly applies to all nonlinear ODEs which are linearizable by point transformations [5,6], such transformations do not alter the dimension of the symmetry algebra.

In this paper, we extend this characterization to equations of higher orders. It turns out that for each equation of order n there will be $n - 2$ characterizing equations, and the limitation of our presentation of the characterizing equations only up to the order five is simply due to their very large size. However, we give a description of the method for deriving this characterization for equations of any order. The derivation of these characterizing equations is also based on the canonical normal form of linear equations admitting a maximal symmetry algebra that was obtained in [5] from a symmetry approach, and in [7] from an iterative approach. These characterizing equations therefore also represent a generalization of the results of [5,7]. We then deduce that the most general form of a linear equation admitting a maximal symmetry algebra can be expressed in

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standard form in terms of only two arbitrary functions. We also deduce that the Laguerre–Forsyth form of a linear equation reduces to the canonical form if and only if the equation has maximal symmetries.

Although we do not give the characterizing equations for each linear equation of order n, we note however that among the $n-2$ characterizing equations exactly one of them represents a semi-invariant of the equation, that is a function of the coefficients of the equation whose expression does not change when the dependent variable is transformed. We obtain an expression for these semi-invariants for equations of all orders and describe some of their properties.

Finally, using some simplifying assumptions and the method of [8], we give expressions for both the symmetry generator X_n of G_S and X_n^0 of the induced pseudo group of transformations G_c, and for all three most common canonical forms of linear equations of a general order n. Here, G_S denotes the symmetry group of the general linear equation in which the arbitrary functions are considered as additional dependent variables.

2. Coefficient characterization

A method based on a symmetry approach has been proposed in [5] for characterizing the coefficients of linear ordinary differential equations (ODEs) that admit a maximal symmetry algebra, but only for equations in reduced normal form (in which the term of second highest order vanishes). In a more recent paper [7] a similar characterization based on an iterative approach was proposed, in which according to a result of Krause and Michel [2] a linear equation admitting a maximal symmetry is simply viewed as an iterative equation. By iterative equation, we mean an equation of the form

$$\Psi^n[y] = 0, \quad y = y(x), \quad n \geqslant 1, \tag{2a}$$

where

$$\Psi^1[y] = ry' + sy, \quad \Psi^n[y] = \Psi^{n-1}[\Psi[y]] \tag{2b}$$

and where $r = r(x)$ and $s = s(x)$ are the parameters of the source equation $\Psi^1[y] = 0$. This characterization shows that in its reduced normal form, a general linear equation depends solely on one arbitrary function $a = a(x)$. For equations of orders three to five, the corresponding equations are given as follows:

$$y^{(3)} + ay' + \frac{a'}{2}y = 0, \tag{3a}$$

$$y^{(4)} + ay'' + a'y' + \left(\frac{3}{10}a'' + \frac{9}{100}a^2\right)y = 0, \tag{3b}$$

$$y^{(5)} + ay^{(3)} + \frac{3}{2}a'y'' + \left(\frac{9}{10}a'' + \frac{16}{100}a^2\right)y' + \left(\frac{1}{5}a^{(3)} + \frac{16}{100}aa'\right)y = 0. \tag{3c}$$

However, as a linear equation need not occur in its reduced normal form, but rather in the most general standard form, it is thus useful to obtain the corresponding characterization for equations in standard form. We let the general linear equation be given in standard form as

$$\Delta(x, y_{(n)}; C) \equiv y^{(n)} + c_{n-1}y^{(n-1)} + c_{n-2}y^{(n-2)} + \cdots + c_0 y = 0, \tag{4}$$

where $C = (c_0, \ldots, c_{n-1})$. Suppose that such an equation has a symmetry algebra of maximal dimension and let its corresponding reduced normal form be given by

$$y^{(n)} + B_{n-2}y^{(n-2)} + B_{n-3}y^{(n-3)} + \cdots + B_0 y = 0, \tag{5}$$

where the B_j for $j = 0, \ldots, n-2$ are its coefficients and depend as already noted above on a single arbitrary function $a = B_{n-2}$ and its derivatives. Let

$$y^{(n)} + A_{n-1}y^{(n-1)} + A_{n-2}y^{(n-2)} + \cdots + A_0 y = 0 \tag{6}$$

be the corresponding standard form of (5), which may be obtained by a transformation of the form

$$y \mapsto y e^{-\frac{1}{n}\int_{x_0}^{x} A_{n-1}\,dx}. \tag{7}$$

Then (4) and (6) must be identical, and in particular the nonzero coefficient A_{n-1} introduced by the transformation (7) satisfies $A_{n-1} = c_{n-1}$, and more generally we have

$$c_j = A_j, \quad \text{for } j = 0, \ldots, n-1. \tag{8}$$

Note that the coefficients c_j in (4) are mere symbols and we wish to find a relationship among them. Given that in (5) the function B_{n-2} is precisely the arbitrary function $a(x)$ labeling the equation, it can be shown by a recursive procedure, or even by induction on n that

$$A_{n-2} = a + \frac{n-1}{2n}c_{n-1}^2 + \frac{n-2}{2}c'_{n-1}.$$