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On compactons induced by a non-convex convection

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ABSTRACT

Using the model equation: $u_t \pm (u^3 - u^2)_x + (u^3)_{xxx} = 0$ we study the impact of a non-convex convection on formation of compactons. In the $(+)$ version, both traveling and stationary compactons are observed, whereas in the $(-)$ branch, compactons may form only for a bounded range of velocities. Depending on their relative speed, interaction of compactons may be close to being elastic or a fission process wherein the collision begets additional compactons.

1. Introduction

Arguably, the starting point of nonlinear science as is known today could in earnest be attached to the discovery of solitons and the underlying mathematical machinery. Apart of their ubiquity and utility in a plethora of physical processes, solitons have greatly enlarged the vocabulary of basic concepts in science. Yet solitons, as is the case with many scientific paradigms, suffer from a number of obvious shortcomings. For one, the nuisance of their infinite tails follows their trail, as it does in almost all other models in mathematical physics, say, Gaussians in diffusive processes. Another feature of solitons concerns their considerably reduced ubiquity in higher spatial dimensions. This can be easily understood though not as easily addressed; typical soliton is due to a dynamic balance between nonlinear force like convection which acts to sharpen the pulse and dispersion which acts to disperse it. However, while the impact of typical nonlinearity does not change much with spatial dimension, the spreading effectiveness of dispersive mechanisms increases. Thus a well-balanced one-dimensional model becomes less balanced in higher dimensions. This in general greatly reduces the prevalence of robust solitonic structures in higher spatial dimensions.

Compactons, solitons with a compact support, [1–6], address both issues. Their finite span makes the dispersive spread almost independent of dimension. Since, mathematically speaking, at the edge of their support compactons cease to be analytical, compacton supporting equations have to be endowed with a mechanism which induces a singular manifold at, say, the ground state and causes a local loss of uniqueness which enables to 'glue' the solution to its trivial extension, to form a compact entity. Clearly, at the connecting point the 'glued' solution has only a finite degree of smoothness. Since the introduction of compactons over twenty years ago via the $K(m,n)$ model equation, [1,2],

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m,n > 1, \tag{1}$$

the subject has undergo both one- and multi-dimensional extensions, covering a wide variety of topics of both theoretical and practical interest, [1–6].

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In the present work we explore the impact of two prototypical types of a non-convex convection on a formation of compact patterns. This is done via the simple model

$$u_t \pm (u^3 - u^2)_x + (u^3)_{xxx} = 0. \tag{2}$$

Cubic dispersion is used for simplicity of presentation. Alternative forms of nonlinear dispersions are presented in Appendix A. Note that whereas the $K(3,3)$ has four local conserved quantities [1]: $\int u dx$, $\int u^4 dx$, $\int u \cos x dx$ and $\int u \sin x dx$, Eq. (2) inherits from $K(3,3)$ only two conserved quantities: $\int u dx$ and $\int u^4 dx$.

The non-convex convection assumed may be due to two opposing convection forces as may occur, for instance, in a liquid layer on a tilted plane when gravity and Marangoni forces act and induce fluid's convection but in opposite directions, c.f., [7], with each being dominant in a different fluid regime, making it imperative to keep both. Such opposing convection mechanisms may lead to the emergence of undercompressive shocks investigated there.

It is natural to examine first the impact of a non-convex convection on familiar grounds of the KdV universe

$$u_t \pm \left(u^3 - u^2\right)_x + u_{xxx} = 0. \tag{3}$$

(Though Eq. (3) may formally be mapped into two mKdVs on a nonzero plateau this will only mask the non-convex features because the relevant background remains trivial). The first thing to note is that each of the assumed convection forms begets a very different pattern. The solitary waves of the $(+)$ branch of convection are

$$u = \frac{3\lambda}{\pm\sqrt{1 + \frac{9\lambda}{2}}\cosh\left(\sqrt{\lambda}(x - \lambda t)\right) - 1} \tag{4}$$

and for a large velocity they converge to the familiar mKdV soliton

$$u = \pm\frac{\sqrt{2\lambda}}{\cosh\left(\sqrt{\lambda}(x - \lambda t)\right)}. \tag{5}$$

As $\lambda \downarrow 0$, a nontrivial stationary soliton emerges

$$u = \frac{12}{2x^2 + 9} \tag{6}$$

which unlike its traveling siblings decays algebraically. This is a non-generic structure which often is referred to as an end-point of a resonant eigenstate with energy sitting at the edge of the continuous spectrum. Here the response to perturbation may depend on the nature of the perturbation. The peculiar state of the stationary entity could be also seen noting that perturbation theory predicts both KdV and mKdV solitons to be stable [8], modulo a minor shift in their speed. However, such seemingly innocuous change is disastrous for the stationary entity because in order to move, it has to change of shape from an algebraic entity into *cosh*. Note that in terms of $v = -u$, Eq. (3) states

$$v_t \pm \left(v^3 + v^2\right)_x + v_{xxx} = 0 \tag{7}$$

and represents a simple amalgam of the KdV-mKdV, as are the resulting solutions. The interesting action occurs close to the threshold amplitude below which solitons cannot form – a consequence of convection's changing role at small amplitudes where it does not cooperate with dispersion to form permanent structures.

The $(-)$ branch of Eq. (3) begets

$$u = \frac{3\lambda}{\sqrt{1 - \frac{9\lambda}{2}}\cosh\left(\sqrt{\lambda}(x - \lambda t)\right) + 1}, \tag{8}$$

provided that $0 < \lambda < 2/9$. Here solution exists only for a bounded range of velocities above which the defocusing effects overtake.

It is the purpose of this paper to investigate Eq. (2). We first show the existence of two families of compacton solutions, namely, p- and n-compactons. Next, Eq. (2) is studied numerically to investigate the dynamics of these compacton solutions and their interactions.

2. Compacton solutions for Eq. (2)

Returning to the core of our problem, we start with the $(+)$ branch of Eq. (2) and for a future reference record the shape of the $K(3,3)$ compactons, see Eq. (1). If

$$s = x - x_0 - \lambda t,$$

then

$$u = \begin{cases} \pm\sqrt{\frac{3\lambda}{2}}\cos\left(\frac{s}{3}\right) & |s| \leqslant \frac{3\pi}{2}, \\ 0, & \text{elsewhere}. \end{cases} \tag{9}$$