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On the chaotic behavior of non-flat billiards ☆

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ABSTRACT

We study the problem of the motion of a particle on a non-flat billiard. The particle is subject to the gravity and to a small amplitude periodic (or almost periodic) forcing and is reflected with respect to the normal axis when it hits the boundary of the billiard. We prove that the unperturbed problem has an impact homoclinic orbit and give a Melnikov type condition so that the perturbed problem exhibit chaotic behavior in the sense of Smale's horseshoe.

1. Introduction

Impact conditions naturally appear in several interesting mechanical systems. For example an inverted pendulum impacting on rigid walls under external periodic excitation is studied in [8], a Duffing vibro-impact oscillator in [16] and other interesting impact models emerge from understanding the dynamics of rigid blocks [11,14]. Many more stimulating examples of impact oscillators are given in books [3,4,6,9,12,13] where different numerical and analytical methods are described to study their dynamics.

Besides the above examples, there is a broad variety of impact systems represented by billiards. A billiard is essentially given by a convex domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary and a particle on it whose motion follows the usual Newton laws of dynamics until it reaches the boundary of Ω at which points it is reflected in the opposite direction with respect to the normal to the boundary at that point, keeping the same scalar velocity. Of course we only consider trajectories hitting the boundary of Ω at its regular points. The theory of flat billiards is by now classical and very well developed. We refer the reader to [5] for more details and references. However, other kinds of billiards are also studied. According to [10], for example, a billiard in a broad sense is the geodesic flow on a Riemannian manifold with boundary.

In this paper we consider such a different kind of billiards: the dynamics of the particle evolves on a surface in $\mathbb{R}^3$, it has unitary mass and is subject to the gravity and an almost periodic forcing alone. Of course, as on alternative view, such a dynamics may also model a particle moving on a flat billiard immersed in a magnetic field.

The surface is described by a graph $z = f(x,y)$ of a function $f \in C^5(\mathbb{R}^2,\mathbb{R}), f(x,y) \geqslant 0$ and $(x,y) \in \overline{\Omega}$. The particle is forced to remain in the surface in the sense that, each time it hits the boundary of $\mathcal{S} := \{(x,y,z) | (x,y) \in \Omega, z = f(x,y)\}$ it is reflected in

☆ This paper is dedicated to Judita whose light has been suddenly switched off at the early age of 23 years.

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the opposite direction with respect to the normal. By normal here we mean a vector $\vec{n}$ in the tangent plane to $\mathcal{S}$ which is orthogonal to the tangent vector to $\partial\mathcal{S}$ at the point of $\partial\mathcal{S}$. To be more precise, suppose $(x(s), y(s), f(x(s), y(s)))$ is a parametric representation of $\partial\mathcal{S}$ then $\vec{n}$ is orthogonal to the tangent vector to $\partial\mathcal{S} : \vec{T} = (x'(s), y'(s), x'(s)f_x(x(s), y(s)) + y'(s)f_y(x(s), y(s)))$ and to the normal vector to the surface $z = f(x, y) : \vec{B} = (-f_x(x(s), y(s)), -f_y(x(s), y(s)), 1)$. So $\vec{n} = \vec{B} \wedge \vec{T}$. For example if, as we assume in this paper, $f(x, y) = 0$ in a neighborhood of the boundary $\partial\Omega$, then:

$$\vec{n} = \begin{pmatrix} -y'(s) \\ x'(s) \\ 0 \end{pmatrix}.$$

Using D'Alembert principle the equation of motion of the particle without an almost periodic forcing, in $\mathcal{S} \setminus \partial\mathcal{S}$ is given by [7, p. 662]

$$\begin{aligned} \ddot{x} &= \lambda f_x(x, y) \\ \ddot{y} &= \lambda f_y(x, y) \\ \ddot{z} &= -\lambda - g \end{aligned} \tag{1.1}$$

where g is the gravitation constant. The constraint $z(t) = f(x(t), y(t))$ and Eq. (1.1) give

$$-\lambda - g = \ddot{z} = \lambda f_x(x, y)^2 + f_{xx}\dot{x}^2 + 2f_{xy}(x, y)\dot{x}\dot{y} + f_{yy}(x, y)\dot{y}^2 + \lambda f_y(x, y)^2,$$

which implies

$$\lambda = -\frac{g + \left\langle H_f(x, y)\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right\rangle}{1 + \|\nabla f\|^2}, \tag{1.2}$$

where ∇f and H_f is the gradient and Hessian of f, respectively. We note that $\nabla f = f'^*$, which we use several times in our paper. As a consequence the problem is reduced to study the behavior of solutions of an almost periodic perturbation of the following unperturbed differential equation on $\Omega = \{(x, y) | x \geqslant 0, 0 \leqslant y \leqslant x \tan\beta\}$:

$$\begin{aligned} \ddot{x} &= \lambda f_x(x, y) \\ \ddot{y} &= \lambda f_y(x, y) \end{aligned} \tag{1.3}$$

where $\lambda = \lambda(x, y, \dot{x}, \dot{y})$ is as in (1.2), and $z = f(x, y)$, together with the requirement that, when $(x(t), y(t)) \in \partial\Omega$ then $(\dot{x}(t), \dot{y}(t))$ is reflected with respect to the normal to $\partial\Omega$ at $(x(t), y(t))$. Hence the solution of (1.1) is forced to remain in Ω.

We emphasize that the main purpose of this paper is to introduce a new class of impact systems modeled by nonlinear billiards with chaotic behavior. So instead of a gravitational force, we could consider other force fields acting on the particle under which it is moving inside Ω. In this paper, we consider the gravitational field since we think that this problem is interesting itself and in addition, it is rather sophisticated for showing all difficulties of technical computations and theoretical background.

To continue, given the messy nature of Eq. (1.2), we assume

$$f(x, y) = F(x - a)^2 + (y - b)^2 \tag{1.4}$$

with $a > 0, 0 < b < a\tan\beta, 0 < \beta < \frac{\pi}{2}$, and F is a C^5 function in $[0, \infty)$ whose support is contained in an interval $[0, r_0^2]$ with $r_0 > 0$ sufficiently small that the closed ball $\overline{B((a, b), r_0)}$ is contained in $\mathring{\Omega}$ and such that $F' \leqslant 0$ with $F'(0) < 0$.

Example 1.1. For illustration, as a concrete example, we take $a = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, b = \sin\frac{\pi}{6} = \frac{1}{2}, \beta = \frac{\pi}{3}$ and $F(r) = (1 - 16r)^6$ for $0 \leqslant r \leqslant r_0^2 = \frac{1}{16}$ and $F(r) = 0$ for $r \geqslant \frac{1}{16}$ (see Fig. 1).

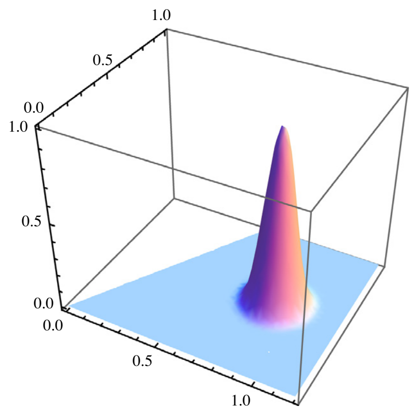

Fig. 1. The graph of $f(x, y)$ in this concrete case on $0 \leqslant x \leqslant 1.2$ and $0 \leqslant y \leqslant \min\{\sqrt{3}x, 1\}$.