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Numerical simulations and modeling for stochastic biological systems with jumps

Xiaoling Zou [a], Ke Wang [a,b,*]

[a] Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai 264209, PR China
[b] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, PR China

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ABSTRACT

This paper gives a numerical method to simulate sample paths for stochastic differential equations (SDEs) driven by Poisson random measures. It provides us a new approach to simulate systems with jumps from a different angle. The driving Poisson random measures are assumed to be generated by stationary Poisson point processes instead of Lévy processes. Methods provided in this paper can be used to simulate SDEs with Lévy noise approximately. The simulation is divided into two parts: the part of jumping integration is based on definition without approximation while the continuous part is based on some classical approaches. Biological explanations for stochastic integrations with jumps are motivated by several numerical simulations. How to model biological systems with jumps is showed in this paper. Moreover, method of choosing integrands and stationary Poisson point processes in jumping integrations for biological models are obtained. In addition, results are illustrated through some examples and numerical simulations. For some examples, earthquake is chose as a jumping source which causes jumps on the size of biological population.

1. Introduction

Systems in nature sciences are inevitably subjected to various environmental noise [1–3]. The classical stochastic biological model is that with a continuous driving process (i.e. a geometric Brownian motion). However, biological systems may suffer sudden environmental shocks: such as earthquakes, floods, toxic pollutants, hurricanes and so on. These sudden environmental shocks will cause jumps on population dynamics. Here, a jump means a sudden shift on the size of biological population, and the mathematical explanation is that sample paths are not continuous almost surely. Thereby, to model these practical phenomena, scholars would like to represent biological population by a process that has jumps, and use stochastic differential equations (SDEs) with jumps to explain systems underlying randomness contain jumps. Theory of SDEs with jumps has been researched widely, such as monographs [4–6] and references cited therein.

Now, we introduce two important types of SDEs that involve jumps. A widely used discontinuous driving process in SDEs is the Lévy process. The first type is SDEs with respect to Lévy processes. Suppose $Z(t)$ is a Lévy process, $B(t)$ is a standard 1-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with a filtration satisfying usual conditions [7–9]. Suppose the drifting coefficient, diffusion coefficient and jumping coefficient are denoted by $b(\cdot), \sigma(\cdot)$ and $f(\cdot)$. A simple Itô-jumps process with respect to Lévy process $Z(t)$ is

$$\mathrm{d}x(t) = b(x(t^-))\mathrm{d}t + \sigma(x(t^-))\mathrm{d}B(t) + f(x(t^-))\mathrm{d}Z(t). \tag{1.1}$$

* Corresponding author at: Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai 264209, PR China. Tel.: +86 06315687086.
E-mail addresses: zouxiaoling1025@126.com (X. Zou), wangke@hitwh.edu.cn (K. Wang).

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For this model, if Lévy process $Z(t)$ has a jump of size z, then $x(t)$ will have a jump of size $f(x(t^-))z$. This is to say $x(t)$ has a jump whose size depends linear on z. However, one might hope $x(t)$ to have a jump depends on z, but not linear in z. So, Eq. (1.1) is not a very good model in many applications. This defect has been pointed out by some authors, see [10,11] for details.

To make more versatile SDEs with jumps, scholars [10,11] introduced the second type of SDEs with jumps, i.e. SDEs driven by compensated Poisson random measures. Let $N(t,A)$ be a Poisson random measure which is often used to describe jumps of stochastic process. Characteristic measure of $N(t,A)$ is denoted by $\lambda(A)$, and compensated Poisson random measure is defined by $\widetilde{N}(t,A) = N(t,A) - t\lambda(A)$. A SDE driven by the compensated Poisson random measure $\widetilde{N}(t,A)$ has the following form:

$$dx(t) = b(x(t^-))dt + \sigma(x(t^-))dB(t) + \int f(x(t^-),z)\widetilde{N}(dt,dz). \tag{1.2}$$

This type of SDEs with jumps can solve the defect of linear dependency mentioned in the last paragraph. So, we will use the second type of SDEs with jumps in this paper.

For SDEs driven by Lévy processes, some scholars introduced numerical schemes for the approximate expectation $\mathbb{E}[f(x(t))]$ for a smooth enough function f, where $x(t)$ is the solution of a SDE driven by a Lévy process, such as [12–15]. Other scholars computed approximate laws for some functions of the sample paths, such as [16]. Rubenthaler [17] researched the numerical simulation of a SDE driven by a Lévy process. He ignored small jumps and assumed the Lévy process can be simulated conditionally outside any given open interval containing zero.

As far as we known, almost all of numerical schemes for SDEs with jumps are focused on how to simulate the Lévy process. This is to say most of numerical schemes for SDEs with jumps are given for the first type of SDEs with jumps. The real difficulty for this is that one cannot know how to simulate a given Lévy process Z_t for all t [17]. In this paper, we will simulate SDEs with jumps from a different angle. We will use the second type of SDEs with jumps to do the simulation, i.e. SDEs driven by compensated Poisson random measures. Rong Situ [4] (see page 32) said "the stochastic jumps perturbation in a dynamical system usually can be modeled as a stochastic integral with respect to some point process, i.e. its counting measure or martingale measure". He studied theory of SDEs with jumps and applications in monograph [4] by using a stationary Poisson point process as the driving process. So, we also assume the compensated Poisson random measure is generated by a stationary Poisson point process in this paper, and show stochastic integration driven by a stationary Poisson point process can be simulated through definition without any neglect.

Compensated Poisson random measure may also be generated by a Lévy process. Bao et al. [18,19] studied asymptotic behaviors for two ecosystems driven by compensated Poisson random measures which are generated by Lévy noise. Kunita [20] treated Itô-jumps SDEs driven by Poisson random measures and Brownian motions, and these two classes of driving terms are obtained from Lévy–Itô decomposition of Lévy processes. The reason why we use a stationary Poisson point process is that stationary Poisson point process can describe where jumps occur and what size jumps are directly [4]. From the biological modeling point of view, for an example, taking earthquake as a jumping source, people always care how many times the magnitude of an earthquake exceeds some level (It is just because that influence of earthquake on biological systems can be ignored when the earthquake is small enough) or when the earthquake happened [4]. The second reason is that we can construct a stationary Poisson point process from a given Lévy process such that they have the same Poisson random measure. i.e. we can use SDE driven by a stationary Poisson point process to simulate the corresponding SDE driven by a Lévy process approximatively. So, assumption that the compensated Poisson random measure is generated by a stationary Poisson point process in this paper is generally. Motivated by the aforesaid works, we will give a numerical method to simulate sample paths for SDEs driven by stationary Poisson point processes, and show the biological significance from mathematical modeling point of view.

Let $x(t^-)$ represents the left limit of $x(t)$, $\xi(t,\omega)$ represents a stationary $\mathfrak{F}_t$-adapted Poisson point process of class QL (Quasi Left-continuous, see page 32 of [4]). Denote $\mathbb{Z} = \mathbb{R} - \{0\}$, A is a Borel set in $\mathbb{Z}$. $N(t,A)$ is a Poisson random measure generated by $\xi(t,\omega)$, $\mathfrak{B}(\mathbb{Z})$ is the Borel σ-algebra on $\mathbb{Z}$. Characteristic measure of $N(t,A)$ is denoted by $\lambda(A)$, where $\lambda(\cdot)$ is a σ-finite measure defined on the measurable space $(\mathbb{Z},\mathfrak{B}(\mathbb{Z}))$ such that $\lambda(\mathbb{Z}) < \infty$. The compensated Poisson random measure is defined by $\widetilde{N}(t,A) = N(t,A) - t\lambda(A)$. Let $b(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ represent the drifting, diffusion and jumping coefficients respectively. For the theory study of SDEs driven by stationary Poisson point processes, a general type in 1-dimensional Euclidean space has the following form:

$$dx(t) = b(t,x(t^-))dt + \sigma(t,x(t^-))dB(t) + \int_{\mathbb{Z}} c(t,x(t^-),z)\widetilde{N}(dt,dz). \tag{1.3}$$

Now, consider the following SDE driven by the Poisson random measure $N(t,A)$ in 1-dimensional Euclidean space:

$$dx(t) = b(t,x(t^-))dt + \sigma(t,x(t^-))dB(t) + \int_{\mathbb{Z}} c(t,x(t^-),z)N(dt,dz), \tag{1.4}$$

by the fact that $N(dz,dt) = \widetilde{N}(dz,dt) + \lambda(dz)dt$, Eq. (1.4) becomes:

$$dx(t) = \left[b(t,x(t^-)) + \int_{\mathbb{Z}} c(t,x(t^-),z)\lambda(dz)\right]dt + \sigma(t,x(t^-))dB(t) + \int_{\mathbb{Z}} c(t,x(t^-),z)\widetilde{N}(dt,dz). \tag{1.5}$$

If we denote