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Differential-algebraic approach to constructing representations of commuting differentiations in functional spaces and its application to nonlinear integrable dynamical systems

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ABSTRACT

There is developed a differential-algebraic approach to studying the representations of commuting differentiations in functional differential rings under nonlinear differential constraints. An example of the differential ideal with the only one conserved quantity is analyzed in detail, the corresponding Lax type representations of differentiations are constructed for an infinite hierarchy of nonlinear dynamical systems of the Burgers and Korteweg–de Vries type. A related infinite bi-Hamiltonian hierarchy of Lax type dynamical systems is constructed.

1. Introduction

We consider the ring $\mathcal{K} := \mathbb{R}\{\{x,t\}\}, (x,t) \in \mathbb{R}^2$, of convergent germs of real-valued smooth Schwartz type functions from $S(\mathbb{R}^2;\mathbb{R})$ and construct the associated differential quotient ring $\mathcal{K}\{u\} := Quot(\mathcal{K}[\Theta u])$ with respect to a functional variable $u \in \mathcal{K}$, where Θ denotes [10,17,4,5,8] the standard monoid of all commuting differentiations D_x and D_t, satisfying the standard Leibniz rule, and defined by the natural conditions

$$D_x(x) = 1 = D_t(t), \quad D_t(x) = 0 = D_x(t), \tag{1.1}$$

The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if the condition $I\{u\} = \Theta I\{u\}$ holds. In the differential ring $\mathcal{K}\{u\}$ the differentiations

$$D_t,\ D_x:\ \mathcal{K}\{u\} \to \mathcal{K}\{u\}, \tag{1.2}$$

satisfy the algebraic commuting relationship

$$[D_t,\ D_x] = 0. \tag{1.3}$$

For an arbitrarily chosen function $u \in \mathcal{K}$ the only representation of (1.3) in the $\mathcal{K}\{u\}$ is of the form

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$$D_t = \partial/\partial t, D_x = \partial/\partial t, \tag{1.4}$$

being the usual partial differentiations. Nonetheless, if the function $u \in \mathcal{K}$ satisfies some additional nonlinear differential-algebraic constraint $Z[u] := Z(u, D_t u, D_x u, \ldots) = 0$, imposed on the ring $\mathcal{K}\{u\}$ for some element $Z[u] \in \mathcal{K}\{u\}$, other nontrivial representations of the differentiations (1.2) in the corresponding reduced and invariant differential ring $\overline{\mathcal{K}}\{u\} := \mathcal{K}\{u\}|_{Z=0} \subset \mathcal{K}\{u\}$ can exist.

Below we will consider in detail this situation and construct the corresponding representations of the commuting relationship (1.3), which are polynomially dependent on $u \in \mathcal{K}$ and its derivatives with respect to the differentiation D_x. The found representations of commuting differentiations D_t and D_x are interpreted as the corresponding Lax type representations for an infinite hierarchy of nonlinear dynamical systems of Burgers and Korteweg–de Vries type.

Remark 1.1. There are interesting applications [14,1] of the problem above in the case when the differentiations D_t and $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ are defined by means of the Lie algebraic conditions

$$[D_x, D_t] = (D_x u) D_x \tag{1.5}$$

and

$$D_x(x) = 1 = D_t(t), \quad D_t(x) = u, D_x(t) = 0 \tag{1.6}$$

for $(x, t) \in \mathbb{R}^2$ and $u \in \mathcal{K}$.

The corresponding representations of the Lie algebraic relationships (1.5) and (1.6) under the differential constraints $D_t^{N+1} u = 0, N \in \mathbb{Z}_+$, of Riemann type imposed on the ideal $\mathcal{K}\{u\}$, prove to be finite dimensional, being equivalent to their so called Lax type representations, important for the integrability theory [2,12,15,22,1] of nonlinear hydrodynamical systems on functional manifolds.

2. Differential rings with the only conserved quantity constraint

Let us pose the following problem:

Problem 2.1. To describe the possible representations of the differentiations D_t, D_x, in the ring $\overline{\mathcal{K}}\{u\} = K\{u\}|_{Z=0}$, defined by a constraint $Z \in \mathcal{K}\{u\}, u \in \mathcal{K}$, and satisfying the conditions (1.1) and (1.3).

As is easy to observe, for the case of arbitrarily chosen function $u \in \mathcal{K}$ and the representation of (1.3) in the ring $\mathcal{K}\{u\}$ is given by the unique expressions (1.4). Another situation arises if there is some differential constraint $Z[u] := Z(u, D_x u, D_t u, \ldots) = 0, Z[u] \in \mathcal{K}\{u\}$, imposed on the function $u \in \mathcal{K}$. Then one can expect that the commutation condition (1.3), if realized in the constrained ideal $\overline{\mathcal{K}}\{u\} := K\{u\}|_{Z[u]=0}$, will be much more specified and in some cases the corresponding representations may appear to be even finite dimensional. The latter may by of nontrivial interest for some applications in applied sciences, especially when these imposed constraints possess some interesting physical interpretation. To be further more precise, we need to involve here some additional differential-algebraic preliminaries [5–8].

Consider the ring $\mathcal{K}\{u\}, u \in \mathcal{K}$, and the exterior differentiation $d : \mathcal{K}\{u\} \to \Lambda^1(\mathcal{K}\{u\}), \ldots,\ d : \Lambda^p(\mathcal{K}\{u\}) \to \Lambda^{p+1}(\mathcal{K}\{u\})$, $p \in \mathbb{Z}_+$, acting in the freely generated Grassmann algebras $\Lambda(\mathcal{K}\{u\}) = \oplus_{p \in \mathbb{Z}_+} \Lambda^p(\mathcal{K}\{u\})$, where by definition,

$$\begin{aligned} \Lambda^1(\mathcal{K}\{u\}) :=\ & \mathcal{K}\{u\}dx + \mathcal{K}\{u\}dt + \sum_{j,k \in \mathbb{Z}_+} \mathcal{K}\{u\} du^{(j,k)}, u^{(j,k)} := D_t^j D_x^k u, \Lambda^2(\mathcal{K}\{u\}) : \\ & = \mathcal{K}\{u\} d\Lambda^1(\mathcal{K}\{u\}), \ldots, \Lambda^{p+1}(\mathcal{K}\{u\}) := \mathcal{K}\{u\} d\Lambda^p(\mathcal{K}\{u\}), \end{aligned} \tag{2.1}$$

The triple $\mathcal{A} := (\mathcal{K}\{u\}, \Lambda(\mathcal{K}\{u\}), d)$ is called *the Grassmann differential algebra* [8] with generatrix $u \in \mathcal{K}$. In the algebra $\mathcal{A}$ one naturally defines the action of differentiations D_t, D_x and $\partial/\partial u^{(j,k)} : \mathcal{A} \to \mathcal{A}, j, k \in \mathbb{Z}_+$, as follows:

$$\begin{aligned} & D_t u^{(j,k)} = u^{(j+1,k)}, D_x u^{(j,k)} = u^{(j,k+1)}, \\ & D_t du^{(j,k)} = du^{(j+1,k)}, D_x du^{(j,k)} = du^{(j,k+1)} \\ & dP[u] = \sum_{j,k \in \mathbb{Z}_+} (\pm) \partial P[u]/\partial u^{(j,k)} \wedge du^{(j,k)} := P'[u] \wedge du, \end{aligned} \tag{2.2}$$

where the sign $\wedge$ denotes the standard [9] exterior multiplication in the differential Grassmann algebra $\Lambda(\mathcal{K}\{u\})$, and for any $P \in \Lambda(\mathcal{K}\{u\})$ the mapping

$$P'[u] \wedge : \Lambda(\mathcal{K}\{u\}) \to \Lambda(\mathcal{K}\{u\}) \tag{2.3}$$

is a linear differential operator in $\Lambda(\mathcal{K}\{u\})$. The following commutation properties

$$D_x d = d D_x, D_t d = d D_t \tag{2.4}$$

hold in the Grassmann differential algebra $\mathcal{A}$. The following remark [8] is also important.

Remark 2.2. The Lie derivative $L_V : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ with respect to a vector field $V : \mathcal{K}\{u\} \to T(\mathcal{K}\{u\})$, satisfying the condition $L_V : \mathcal{K} \subset \mathcal{K}$, can be uniquely extended to the differentiation $L_V : \mathcal{A} \to \mathcal{A}$, satisfying the commutation condition $L_V d = d\, L_V$.

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