Commun Nonlinear Sci Numer Simulat 18 (2013) 443–452

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Generalized transversality conditions in fractional calculus of variations

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ARTICLE INFO

Article history:
Received 27 December 2011
Received in revised form 3 June 2012
Accepted 15 July 2012
Available online 3 August 2012

Keywords:
Calculus of variations
Fractional calculus
Fractional Euler–Lagrange equation
Transversality conditions
Caputo fractional derivative

ABSTRACT

Problems of calculus of variations with variable endpoints cannot be solved without transversality conditions. Here, we establish such type of conditions for fractional variational problems with the Caputo derivative. We consider: the Bolza-type fractional variational problem, the fractional variational problem with a Lagrangian that may also depend on the unspecified end-point $\varphi(b)$, where $x = \varphi(t)$ is a given curve, and the infinite horizon fractional variational problem.

1. Introduction

The calculus of variations is concerned with the problem of extremizing functionals. It has many applications in physics, geometry, engineering, dynamics, control theory, and economics. The formulation of a problem of the calculus of variations requires two steps: the specification of a performance criterion; and then, the statement of physical constraints that should be satisfied. The basic problem is stated as follows: among all differentiable functions $x : [a, b] \to \mathbb{R}$ such that $x(a) = x_a$ and $x(b) = x_b$, with x_a, x_b fixed reals, find the ones that minimize (or maximize) the functional

$$J(x) = \int_a^b L(t, x(t), x'(t))\, dt.$$

One way to deal with this problem is to solve the second order differential equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'} = 0$$

called the Euler–Lagrange equation. The two given boundary conditions provide sufficient information to determine the two arbitrary constants. But if there are no boundary constraints, then we need to impose another conditions, called the natural boundary conditions (see e.g. [19]),

$$\left[\frac{\partial L}{\partial x'}\right]_{t=a} = 0 \quad \text{and} \quad \left[\frac{\partial L}{\partial x'}\right]_{t=b} = 0. \tag{1}$$

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1007-5704/$ - see front matter
http://dx.doi.org/10.1016/j.cnsns.2012.07.009

Clearly, such terminal conditions are important in models, the optimal control or decision rules are not unique without these conditions.

Fractional calculus deals with derivatives and integrals of a non-integer (real or complex) order. Fractional operators are non-local, therefore they are suitable for constructing models possessing memory effect. They found numerous applications in various fields of science and engineering, as diffusion process, electrical science, electrochemistry, material creep, viscoelasticity, mechanics, control science, electromagnetic theory, etc. Fractional calculus is now recognized as vital mathematical tool to model the behavior and to understand complex systems (see, e.g., [20,25,32,38,39,44,46,52]). Traditional Lagrangian and Hamiltonian mechanics cannot be used with nonconservative forces such as friction. Riewe [51] showed that fractional formalism can be used when treating dissipative problems. By inserting fractional derivatives into the variational integrals he obtained the respective fractional Euler–Lagrange equation, combining both conservative and nonconservative cases. Nowadays the fractional calculus of variations is a subject under strong research. Investigations cover problems depending on Riemann–Liouville fractional derivatives (see, e.g., [1,9,13,17,29]), the Caputo fractional derivative (see, e.g., [2,6,11,12,30,41,42]), the symmetric fractional derivative (see, e.g., [37]), the Jumarie fractional derivative (see, e.g., [7,33,34]), and others [3,5,15,16,24,27,28].

The aim of this paper is to obtain transversality conditions for fractional variational problems with the Caputo derivative. Namely, three types of problems are considered: the first in Bolza form, the second with a Lagrangian depending on the unspecified end-point $\varphi(b)$, where $x = \varphi(t)$ is a given curve, and the third with infinite horizon. We note here, that from the best of our knowledge, fractional variational problems with infinite horizon have not been considered yet, and this is an open research area.

The paper is organized in the following way. Section 2 presents some preliminaries needed in the sequel. Our main results are stated and proved in the remaining sections. In Section 3 we consider the Bolza-type fractional variational problem and develop the transversality conditions in a compact form. As corollaries, we formulate conditions appropriate to various type of variable terminal points. Section 4 provides the necessary optimality conditions for fractional variational problems with a Lagrangian that may also depend on the unspecified end-point $\varphi(b)$, where $x = \varphi(t)$ is a given curve. Finally, in Section 5 we present the transversality condition for the infinite horizon fractional variational problem.

2. Preliminaries

In this section we present a short introduction to the fractional calculus, following [26,36,47]. In the sequel, $\alpha \in (0,1)$ and Γ represents the Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then,

1. the left and right Riemann–Liouville fractional integrals of order α are defined by

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt$$

and

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt,$$

respectively;

2. the left and right Riemann–Liouville fractional derivatives of order α are defined by

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt$$

and

$$ {}_xD_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\, dt,$$

respectively.

Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then,

1. the left and right Caputo fractional derivatives of order α are defined by

$$ {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f'(t)\, dt$$