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Contour integral method for European options with jumps

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ABSTRACT

We develop an efficient method for pricing European options with jump on a single asset. Our approach is based on the combination of two powerful numerical methods, the spectral domain decomposition method and the Laplace transform method. The domain decomposition method divides the original domain into sub-domains where the solution is approximated by using piecewise high order rational interpolants on a Chebyshev grid points. This set of points are suitable for the approximation of the convolution integral using Gauss–Legendre quadrature method. The resulting discrete problem is solved by the numerical inverse Laplace transform using the Bromwich contour integral approach. Through rigorous error analysis, we determine the optimal contour on which the integral is evaluated. The numerical results obtained are compared with those obtained from conventional methods such as Crank–Nicholson and finite difference. The new approach exhibits spectrally accurate results for the evaluation of options and associated Greeks. The proposed method is very efficient in the sense that we can achieve higher order accuracy on a coarse grid, whereas traditional methods would required significantly more time-steps and large number of grid points.

1. Introduction

In the general framework of the Black–Scholes model, the underlying stock price asset follows a geometric Brownian motion process and has a continuous sample path defined by

$$\frac{dS}{S} = \mu dt + \sigma dW_t. \tag{1.1}$$

Here S represents the underlying stock price at time t. It is assumed that the associated sample path is continuous. The constants μ and σ represent the expected return on the stock and the volatility of the return respectively; dW_t is the standard Brownian motion or a Wiener process. The Black–Scholes model predicts that the stock price S follows a log-normal distribution at any future time t, i.e.,

$$S(t) = S_0 e^{\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}.$$

The continuity of the sample path indicates that the stock price can only change by a small amount in short interval. However, the reality on the stock market is different. Jumps are regularly observed in the discrete movement of the stock price $S(t)$. These jumps cannot be capture by the log-normal distribution characteristic of the stock price in the Black–Scholes setting and therefore an alternative model which addresses this shortcoming is necessary.

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To overcome the above mentioned shortcoming, a number of models have been proposed in the literature that more appropriately describe the movement of the stock price in the market. Among these, the jump-diffusion model proposed in [9] by Merton is one of the most widely used model. In this framework, the Brownian motion observed in the Black–Scholes model is combined with a poisson distribution which model the jumps discontinuities that normally occur on the market place. For the jump-diffusion model, the movement of the stock price is therefore modeled by the following stochastic differential equation (SDE)

$$\frac{dS}{S} = (\mu - \lambda\kappa)dt + \sigma dW_t + dq. \tag{1.2}$$

As in the previous model, σ represents the volatility, μ is the instantaneous expected return on the stock, and λ is the intensity of the poisson precess (or the jump arrival rate), dW_t is the increment of the Brownian motion process, $\kappa = E(\eta - 1)$, where E is the expectation and $\eta - 1$ is the impulse producing the jump from S to $S\eta$ if a Poisson event occurs and dq is the independent Poisson process defined by

$$dq = \begin{cases} 0 & \text{with probability} \quad 1 - \lambda dt, \\ 1 & \text{with probability} \quad \lambda dt. \end{cases}$$

Using the Itô formula, the SDE (1.2) is rewritten in the form of the following partial integro-differntial equation (PIDE):

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda\kappa)S\frac{\partial V}{\partial S} - (r + \lambda)V + \lambda\int_0^\infty V(S\eta)\psi(\eta)d\eta. \tag{1.3}$$

In the above, $V(S,t)$ is the value of the option depending on the underlying stock price S at any given time t, T is the expiry date, r is the risk free interest rate ($r \geqslant 0$), λ is the intensity of the Poisson process ($\lambda > 0$), κ is the expected jump size, t is the current time, $\psi(\eta)$ is the probability function of the jump amplitude η, where $\psi(\eta) \geqslant 0$, for all η, and is defined by

$$\psi(\eta) = \frac{e^{-\frac{(\log\eta - \mu)^2}{2\gamma^2}}}{\sqrt{2\pi}\gamma\eta}. \tag{1.4}$$

Note that $\int_0^\infty \psi(\eta)d\eta = 1$, and when $\lambda = 0$ in (1.3), we recover the standard Black–Scholes partial differential equation.

For European options, Merton [9] derived analytical expressions but for most exotic options under jump-diffusion models, no closed-form solutions exist and one needs to find numerical solutions for the partial integro-differential equations that arise. However, the convolution integral (1.3) add to the difficulty of finding efficient numerical solutions. Commonly used finite difference methods (FDMs) hardly attain higher order accuracy [2] and typical quadrature rules such as the trapezoidal and Simpson's rules are of low order compared to Gaussian quadrature. However, the later is expensive to implement since it requires the interpolation to match the Chebyshev grid point with those of the FDMs. To reduce the computational cost in solving the convolution integral term, Fast Fourier Transform (FFT) was used in [2,19].

Tangman et al. [14] proposed a different approach in combining the central difference method and the exponential time differencing (ETD) scheme to solve (1.3). The ETD method was proved as very effective and gave second order accuracy. This successful result, encouraged these authors to apply higher order discrete method such as spectral methods to enhance the spatial convergence of the solution. To get around the non-smooth initial condition, a cluster grid of Chebyshev points at the discontinuous point and at boundaries were performed and they obtained fourth order results.

Spectral method are attractive for their exponential convergence rate. This presents an advantage for a direct computation of the convolution integral by a high order Gauss quadrature method. However, high rate of convergence of the spectral method is only guaranteed for smooth solution, a condition which is not fulfilled for the jump-diffusion model (1.3) which has a non-smooth initial condition.

To overcome this situation, one might consider using a spectral element approach. This is the approach followed in [18] where the PIDE (1.3) is solved and the resulting discrete ODE is integrated in time using Crank-Nicolson method. This resulted in spectrally accurate results in space and second order accuracy in time. The exponential results are partly due to the successful approximation of the integral term by Gauss quadrature rule. However, the application of the spectral element involved successive approximation of different integrals generated by the weak form and hence computationally expensive.

In this paper, we propose the use of a multi-domain spectral method. This method uses the spectral method directly in each sub-domains. Matching conditions are imposed to ensure the continuity of the solution and that of its first derivative. After this spatial discretization, the resulting system of ODEs is solved by the Laplace transformation. To recover the solution, an inversion of the Laplace transform solution is performed using the Talbot's method [13] which is based on the application of trapezoidal rule to approximate a Bromwich integral.

The rest of this paper is organized as follows. In Section 2, we give a description of the jump-diffusion model and derive formula for the resolution of the convolution integral. In Section 3, we describe the spectral domain decomposition method for the differential part as well as the integral part. The later is computed by the Gauss–Legendre quadrature. Section 4 deals with the application of the Laplace transform to solve the semi-discrete problem. We also discuss the error analysis related to this approximation in this section. Section 5 contains the numerical comparisons of results obtained by approach with more conventional methods such as Crank–Nicholson for time integration and finite difference for space discretization. Some concluding remarks and scope for future research are given in Section 6.