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Mathieu equation with application to analysis of dynamic characteristics of resonant inertial sensors

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ABSTRACT

In this paper, Mathieu equation is applied to analyze the dynamic characteristics of resonant inertial sensors. Unlike previous work, Mathieu equation is not just a differential equation and analyzes the stability of the transition curves, but become an important method in analyzing parametric resonant characteristics and approximate output of resonant inertial sensors. It is demonstrated that the mathematical model of resonant inertial sensors is described by Mathieu equation. The relevant Mathieu equation theory and dynamic characteristics analysis methods were proposed, which include both stability and dynamic linear output. Finally, theoretical and experimental analysis show that the correlation of the theoretical curve and the experimental result coincide so perfectly, which means proposed analysis methods for Mathieu equation could be used to analyze the dynamic output characteristic of resonant inertial sensors. The theoretical analyzing approach of Mathieu equation and experimental results of resonant inertial sensors are obtained, which provide an application area for Mathieu equation and a reference for the robust design for resonant inertial sensors.

1. Introduction

The majority of equations used in technical and applied mathematics have originated as the result of investigating practical problems. Mathieu equation was introduced by their originator in 1868 [1], when he determined the vibrational modes of a stretched membrane having an elliptical boundary.

The Mathieu equation

$$q''(z) + (\delta + \varepsilon \cos 2z)q(z) = 0 \tag{1.1}$$

occurs in a wide variety of practical problems, which has been studied in great detail since its discovery. Mathieu equation is an excellent paradigm to study parametric resonance, henceforth referred to as PR, and it exemplifies the response of a system to the kind of periodic force. In fluid dynamics, many examples of waves may be excited by PR. Other examples can be found in acoustic instabilities in flames [2] and Rayleigh–Benard convection [3]. In geophysical fluid dynamics, Poulin et al. [4] and Pedlosky and Thomson [5] derived ordinary differential equations of the Hill type to describe the stability of barotropic and baroclinic shear flows, respectively. A rediscovery of damped Mathieu equation appears in Ref. [6], where it is

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shown that the elastic motion of a Large Space Structure. But the reports about Mathieu equation applied in kinetics sensors have not been detailed described.

The inertial sensors are targeted for accelerometers and gyroscopes. Although the inertial sensors have been the subject of intensive research in the past few years [7–14], they are just researched about novel structure, control strategies, and the working principles of the sensors were not resonant exciting and resonant sensing. Resonant inertial sensors are not only advantageous for micromechanical actuators but also for resonant sensing. Resonant sensing often involves the detection of input measurand by means of resonant frequency shift in the sensing device. Therefore resonant inertial sensors technique benefit from being highly sensitive, have the potential for large dynamic range, good linearity, low noise and potentially low power.

However, there are not too many units taking the research about resonant inertial sensors and their dynamic characteristics. Seshia [15] reported integrated microelectromechanical resonant inertia sensors, but the dynamic characteristics of resonant inertial sensors were not introduced in detail. Moussa and Bourquin [16] reported the theory of direct frequency output vibratory gyroscope, but the gyroscope they researched that the vibratory mode is out of plane, and the detection principle is not resonant sensing to measure the Coriolis force directly. There are some other literatures about resonant inertial sensors, but they are all focusing on driving control and fabricated process [17–22].

The main reason for resonant inertial sensors develops so slowly is that the dynamic characteristics of resonant inertial sensors just are parametrically excited systems, which could be described by differential equations in which the input appears as a time dependent coefficient. They are typified by the Mathieu equation. This work extends Mathieu equation dynamic characteristics analysis to resonant inertial sensors. A series of differential equations are necessary to model the resonator dynamics. The resonant frequency is difficult to model and there is considerable uncertainty in the parameters, so the dynamic characteristics analysis is needed, and Mathieu equation with application to analysis of dynamic characteristics of resonant inertial sensors is necessary. Moreover, Mathieu equation cannot be solved analytically in terms of standard functions, the reason is that one of the coefficients is not constant but time-dependent.

This work using the notion of parametric resonance to create dynamic characteristics analyzing methods for resonant inertial sensors systems. About the analyzing methods, recently several authors have studied the stability of and approximate solution for Mathieu equation. In [23–25], the authors divided the parameter space into regions of stability/instability. Arrowsmith and Mondragon [26] analyzed how the size of parametric regions of stability for the Mathieu equation can be enlarged. In [27–31], the authors analyzed the of quasi-periodic nonlinear Mathieu equation, the extensions to the Mathieu equation contain two periodic terms in different frequencies, which are not applicable in our research.

This work focuses on analyzing of Mathieu equation with application to dynamic characteristics of resonant inertial sensors, enabling the resonant inertial sensors to robust design. Since this is the focus, here the applications of each of the resonant inertial sensors studied are introduced using simple method.

The rest of this paper is organized as follows. Section 2 derive the proposed Mathieu equation relevant theory with particular emphasis on how to analyze dynamic characteristics of resonant inertial sensors. The method is then applied to resonant acceleration sensors and resonant angular rate sensors in Sections 3 and 4 summarizes the conclusions.

2. Theoretical development

2.1. Mathematical modeling of resonant inertial sensors

Resonance is a property of a system to describe an enhanced response at a certain characteristic natural frequency that is wholly determined by parameters of the system [15]. The specific frequency is one where the system retains input energy with minimum loss. At a microscopic scale, operating systems at resonance enhances the effects of small forces and the device signal-to-noise ratio. Resonant sensing has been implemented in numerous devices for the measurement of pressure [32], humidity [33], temperature [34], acceleration [35], mass flow [36], specific gas [37], biological detection (immunosensors [38], cytometers [39]), force (AFM cantilevers [40]) and magnetic field [41]. The resonator sensor element is often built into a large device that transmits the effect of the parameter to be measured as a variation in spring constant of the resonant sensor element. While changing spring constant changes the frequency of the system. The resonant sensor element can take a number of forms such as a cantilever, a double-ended tuning fork or a singly clamped beam.

For the resonant inertial sensors we researched, the functional schematic of a resonant inertial sensor is shown in Fig. 1, the time-varying axial force (acceleration sensor: dynamic acceleration, angular rate sensor: Coriolis force) act on the proof

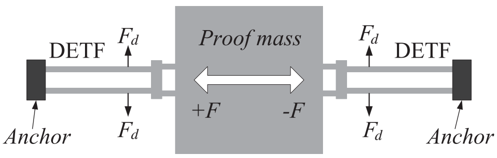

Fig. 1. Functional block diagram schematic of resonant inertial sensors.