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Symmetry analysis and some exact solutions of cylindrically symmetric null fields in general relativity

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ABSTRACT

The symmetry reduction method based on the Fréchet derivative of the differential operators is applied to investigate symmetries of the Field equations in general relativity corresponding to cylindrically symmetric space–time, that is a coupled system of nonlinear partial differential equations of second order. More specifically, this technique yields invariant transformation that reduce the given system of partial differential equations to a system of nonlinear ordinary differential equations. Some of the reduced systems are further studied for exact solutions.

1. Introduction

In general relativity the investigation of null fields has acquired considerable interest in connection with the study of gravitational radiation. In the invariant theory of gravitational radiation, the Riemann tensor plays the central part. The algebraic and differential properties of this tensor have been discussed with a view to characterizing wave fields in general relativity.

The essential idea in the Riemann tensor analysis as applied to radiation theory is that in a gravitational radiation field, the Riemann tensor will lie in some special relationship to the null cone. One such relationship is [1]

$$\left(R_{hijk} + iR^{*}_{hijk}\right)\omega^k = 0, \tag{1.1}$$

where ω^k is a null vector, R_{hijk} is the Riemann tensor and R^{*}_{hijk} is its dual. Eq. (1.1) imposes on the Riemann tensor a very severe restrictions, which is satisfied only asymptotically in the wave zone of a radiating system and exactly in plane gravitational waves and a few other special cases.

From (1.1), it follows that

$$R_{ij} = \sigma\omega_i\omega_j, \tag{1.2}$$

where σ is a scalar. Since ω^i is a null vector, from (1.2) the spur of the Ricci tensor R vanishes identically and through Einstein field equations the energy momentum tensor can be written as

$$-8\Pi T_{ij} = \sigma\omega_i\omega_j. \tag{1.3}$$

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In the view of (1.3) the four eigen values $s_{(i)}$ determined by the equation

$$|T_{ij} - sg_{ij}| = 0$$

turn out to be zero and then the equation $T_{ij}v^j = sg_{ij}v^j$ admits only one real eigenvector which is null. The energy momentum tensor given by (1.3) represents some kind of incoherent superposition of wave packets of radiation and any zero rest mass field can be the source of radiation. We propose, in the present paper, to drive certain solutions of geometrical Eq. (1.2), when $\sigma \neq 0$, for the cylindrical symmetric space time with two degrees of freedom.

1.1. The metric form and the Field equations

We consider the cylindrical symmetric space time [2]

$$ds^2 = \exp(2w - 2u)(dt^2 - dr^2) - (v^2\exp(2u) + r^2\exp(-2u))d\phi^2 - \exp(2u)dz^2 - 2v\exp(2u)d\phi dz, \tag{1.4}$$

where u, v and w are functions of r and t only. When $v = 0$, (1.4) reduces to the well known Einstein Rosen metric with one degree of freedom.

The nonzero components of the Ricci tensor obtained from (1.4) are

$$\begin{aligned}
R_{44} &= -w_{11} - \frac{w_1}{r} + w_{44} + u_{11} + \frac{u_1}{r} - u_{44} + 2u_4^2 + \frac{\exp(4u)}{2r^2}v_4^2,\\
R_{11} &= w_{11} - \frac{w_1}{r} - w_{44} - u_{11} - \frac{u_1}{r} + u_{44} + 2u_4^2 + \frac{\exp(4u)}{2r^2}v_1^2,\\
R_{14} &= -\frac{w_4}{r} + 2u_1u_4 + \frac{\exp(4u)}{2r^2}v_1v_4,\\
R_{33} &= \exp(4u - 2w)\left(u_{11} + \frac{u_1}{r} - u_{44} - \frac{\exp(4u)}{2r^2}(v_1^2 - v_4^2)\right),\\
R_{23} &= vR_{33} + \frac{\exp(4u - 2w)}{2}\left(v_{11} - \frac{v_1}{r} - v_{44} + 4(u_1v_1 - u_4v_4)\right),\\
R_{22} &= 2vR_{23} - (v^2 + r^2\exp(-4u))R_{33}.
\end{aligned} \tag{1.5}$$

Here and in what follows, the subscripts 1 and 4 after u, v and w represent the partial differentiation with respect to r and t respectively.

If the direction of propagation of the wave is the positive r-direction, we have here $\omega^2 = \omega^3 = 0$, $\omega^1 = \omega^4$ and, therefore, from (1.2) and (1.4) we get the field equations

$$R_{11} - R_{44} = 0, \tag{1.6a}$$
$$R_{14} + R_{44} = 0, \tag{1.6b}$$
$$R_{22} = R_{23} = R_{33} = 0. \tag{1.6c}$$

Making use of expressions for R_{ij} given in (1.5), the relations (1.6a)–(1.6c) give the four differential equations

$$u_{11} + \frac{1}{r}u_1 - u_{44} = \frac{1}{2}r^{-2}\exp(4u)(v_1^2 - v_4^2), \tag{1.7}$$
$$v_{11} - \frac{1}{r}v_1 - v_{44} = 4(u_4v_4 - u_1v_1), \tag{1.8}$$
$$w_1 + w_4 - r(u_1 + u_4)^2 = \frac{\exp(4u)}{4r}(v_1 + v_4)^2, \tag{1.9}$$
$$w_{11} - w_{44} + u_1^2 - u_4^2 = \frac{\exp(4u)}{4r^2}(v_1^2 - v_4^2). \tag{1.10}$$

So, we have four Eqs. (1.7)–(1.10) for the determination of three unknowns u, v, and w and one can easily verify that these all are consistent. Therefore, we drop Eq. (1.10) and solve the remaining equations for u, v and w. It may be pointed out that Eqs. (1.7,1.8) are a set of coupled, second order, nonlinear partial differential equations in u and v, hence we will concentrate on these two equations and Eq. (1.9) is first order linear partial differential equation in w, which we will solve once we get u and v. Rewriting the Eqs. (1.7,1.8)

$$\begin{aligned}
u_{rr} + \frac{1}{r}u_r - u_{tt} &= \frac{1}{2}r^{-2}\exp(4u)(v_r^2 - v_t^2),\\
v_{rr} - \frac{1}{r}v_r - v_{tt} &= 4(u_tv_t - u_rv_r).
\end{aligned} \tag{1.11}$$

The system (1.11) also represents another important class of Einstein field equations for vacuum [2]. No systematic investigation of this problem has yet been made (for more details, please refer to [2] pp - 350–351). Examples are the the counterpart of the Kerr solution [3] and of the Tomimatsu–Sato solutions [4]. Due to nonlinearity of exponential order, it is