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On modified method of simplest equation for obtaining exact and approximate solutions of nonlinear PDEs: The role of the simplest equation

Nikolay K. Vitanov *

Institute of Mechanics, Bulgarian Academy of Sciences, Akad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria
Max-Planck-Institute for the Physics of Complex Systems, Noethnitzerstr. 38, 01187 Dresden, Germany

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ABSTRACT

The modified method of simplest equation is powerful tool for obtaining exact and approximate solutions of nonlinear PDEs. These solutions are constructed on the basis of solutions of more simple equations called simplest equations. In this paper we study the role of the simplest equation for the application of the modified method of simplest equation. We follow the idea that each function constructed as polynomial of a solution of a simplest equation is a solution of a class of nonlinear PDEs. We discuss three simplest equations: the equations of Bernoulli and Riccati and the elliptic equation. The applied algorithm is as follows. First a polynomial function is constructed on the basis of a simplest equation. Then we find nonlinear ODEs that have the constructed function as a particular solution. Finally we obtain nonlinear PDEs that by means of the traveling-wave ansatz can be reduced to the above ODEs. By means of this algorithm we make a first step towards identification of the above-mentioned classes of nonlinear PDEs.

1. Introduction

In the last two decades the nonlinear models of natural and social phenomena become dominant in science [1–7]. Because of this the research on the nonlinear partial differential equations increased and now the NPDEs are widely applied in the theory of solitons [8], in the mathematical social dynamics [9,10], biology [11], theory of dynamical systems, chaos theory and ecology [12–16], hydrodynamics and theory of turbulence [17–24], etc.

Many of the model systems of NPDEs are large and are accompanied by complex boundary conditions. For such systems one tries to obtain numerical solutions. But in addition to the numerical solutions of the corresponding model equations it is of great interest to obtain exact analytical solutions of simple or more complex NPDEs and systems of NPDEs. Such exact solutions describe important classes of waves and processes in the investigated systems. Moreover the exact solutions can be used to test the computer programs for obtaining numerical solutions of the corresponding nonlinear PDEs. Finally the exact solutions can be useful as initial conditions in the process of obtaining of numerical solutions.

Because of all above an important research area is connected to obtaining exact analytical or approximate numerical solutions of such model PDEs. The inverse scattering transform [25–28], and the method of Hirota [29] are famous methods for obtaining exact soliton solutions of various NPDEs. In addition in the last several years several approaches for obtaining exact special solutions of nonlinear PDE have been developed (see for examples [30–32]). By means of such methods numerous exact solutions of many equations have been obtained such as for an example the Kuramoto–Shivasinsky equation [31,33], equations, connected to the models of population dynamics [34–44], sine-Gordon equation [45–52], sinh-Gordon or Poisson–Boltzmann equation [53–55], Lorenz – like systems [56], or water waves [57–61].

* Address: Institute of Mechanics, Bulgarian Academy of Sciences, Akad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria.
E-mail addresses: vitanov@imbm.bas.bg, vitanov@mpipks-dresden.mpg.de

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The discussion below will be concentrated around the modified method of simplest equation for obtaining exact and approximate solutions of nonlinear PDEs. The method of simplest equation has been developed by Kudryashov et al. [33,43,62–65] on the basis of a procedure analogous to the first step of the test for the Painleve property [66]. In the modified method of the simplest equation [31,32,44] this procedure is substituted by the concept for the balance equation. Modified method of simplest equation is already successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of generalized Kuramoto–Sivashinsky equation, reaction–diffusion equation, reaction-telegraph equation [31,41] generalized Swift–Hohenberg equation and generalized Rayleigh equation [32], generalized Fisher equation, generalized Huxley equation [44], generalized Degasperis–Processi equation and b-equation [67].

In more details the discussion below will be devoted to the role of the simplest equation for the implementation of the modified method of simplest equation. In 2004 Kudryashov [68] used the equation for the Weierstrass elliptic function as building block to find a number of differential equations with exact solutions. Below we follow this idea and use three equations: the equations of Bernoulli, Riccati and the elliptic equation as building blocks to find classes of equations with exact solutions.

The organization of the paper is as follows. In Section 2 we give a brief description of the modified method of simplest equation and show the importance of the kind of the used simplest equation. In the following sections we discuss solutions of nonlinear ODEs and PDEs based on three kinds of simplest equations: Riccati equation in Section 3, Bernoulli equation in Section 4, and the elliptic equation in Section 5. Several concluding remarks are summarized in Section 6.

2. The modified method of simplest equation

2.1. Brief description of the method

Let us have a partial differential equation and let by means of an appropriate ansatz this equation be reduced to a nonlinear ordinary differential equation

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots\right) = 0 \tag{2.1}$$

For large class of equations from the kind (2.1) exact solution can be constructed as finite series

$$F(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [\Phi(\xi)]^\mu \tag{2.2}$$

where $\nu > 0$, $\mu > 0$, p_μ are parameters and $\Phi(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation. The simplest equation is of lesser order than (2.1) and we know the general solution of the simplest equation or we know at least exact analytical particular solution(s) of the simplest equation [62,63].

The modified method of simplest equation can be applied to nonlinear partial differential equations of the kind

$$E\left(\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}}, \frac{\partial^{\omega_2}F}{\partial t^{\omega_2}}, \frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}}\right) = G(F) \tag{2.3}$$

where $\omega_3 = \omega_4 + \omega_5$ and

1. $\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}}$ denotes the set of derivatives
$$\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}} = \left(\frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial F^3}{\partial x^3}, \dots\right)$$
2. $\frac{\partial^{\omega_2}F}{\partial t^{\omega_2}}$ denotes the set of derivatives
$$\frac{\partial^{\omega_2}F}{\partial t^{\omega_2}} = \left(\frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t^2}, \frac{\partial F^3}{\partial t^3}, \dots\right)$$
3. $\frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}}$ denotes the set of derivatives
$$\frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}} = \left(\frac{\partial^2 F}{\partial x\partial t}, \frac{\partial^3 F}{\partial x^2\partial t}, \frac{\partial F^3}{\partial x\partial t^2}, \dots\right)$$
4. $G(F)$ can be
   (a) polynomial of F or,
   (b) function of F which can be reduced to polynomial of F by means of Taylor series for small values of F.
5. E can be an arbitrary sum of products of arbitrary number of its arguments. Each argument in each product can have arbitrary power. Each of the products can be multiplied by a function of F which can be