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ℓ^∞-stability analysis of discrete autonomous systems described by Laurent polynomial matrix operators

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ABSTRACT

In this paper, we analyze the ℓ^∞-stability of infinite dimensional discrete autonomous systems, whose dynamics is governed by a Laurent polynomial matrix $A(\sigma, \sigma^{-1})$ in shift operator σ on vector valued sequences. We give necessary and sufficient conditions for the ℓ^∞-stability of such systems. We also give easy to check tests to conclude or to rule out the ℓ^∞-stability of such systems.

1. Introduction

Infinite dimensional systems – that is, dynamical systems defined over an infinite dimensional state-space – arise as a natural mathematical model for numerous engineering applications. In fact, any system that is modeled by partial differential/difference equations (distributed parameter systems) or by delay-differential equations can be cast as an infinite dimensional dynamical system [1]. Naturally, the question of stability of such systems is an important issue. However, owing to the infinite dimensionality of the state-space, extension of results on stability of finite dimensional systems is often not possible. The question of stability of a certain special class of infinite dimensional systems has been dealt with in the recent interesting work of Feintuch and Francis [2] concerning an infinite chain of vehicles. In [2], the dynamics of the infinite chain of vehicles follows the *nearest-neighbor interaction*: let $q_n(t)$ denote the position of the nth vehicle at time t, then

$$\dot{q}_n = f(q_{n+1} - q_n, q_{n-1} - q_n),$$

where f is the same linear function for all n. Note that, such a dynamical equation can be written succinctly as:

$$\dot{\mathbf{q}} = (a_{-1}\sigma^{-1} + a_0 + a_1\sigma)\mathbf{q}, \tag{1}$$

where $\mathbf{q}$ denotes the entire sequence $\{\ldots, q_{-1}, q_0, q_1, \ldots\}$ and σ is the (left or right) shift operator with a_{-1}, a_0, a_1 being real numbers. The operator $(a_{-1}\sigma^{-1} + a_0 + a_1\sigma)$ has the structure of a Laurent polynomial operator in the shift σ. In this paper, we deal with stability of dynamical systems whose dynamics is governed by a generalized discrete version of (1): while (1) involves only scalar trajectories, we consider vector trajectories and instead of just nearest-neighbor interactions, we consider an operator given by a general Laurent polynomial matrix. Thus, the systems we are concerned with are governed by the following type of discrete dynamical equation:

$$\mathbf{x}_{k+1}(\cdot) = A(\sigma, \sigma^{-1})\,\mathbf{x}_k(\cdot), \tag{2}$$

where $A(\sigma, \sigma^{-1})$ is a square Laurent polynomial matrix in shift operator σ, and $\mathbf{x}_k(\cdot)$ is a vector valued sequence defined over integers.

Unlike its finite dimensional counter-part, stability analysis of infinite dimensional systems depends crucially on the normed space chosen as the infinite dimensional state-space. The two most prevalent normed spaces in this regard are $(\ell^2, \|\cdot\|_2)$ and $(\ell^\infty, \|\cdot\|_\infty)$. While working with $(\ell^2, \|\cdot\|_2)$ space is somewhat easier than with $(\ell^\infty, \|\cdot\|_\infty)$ space, in many questions of practical significance, it is $(\ell^\infty, \|\cdot\|_\infty)$ space that becomes the more realistic choice. For example, in the case of infinite chain of vehicles, ℓ^2 perturbation from an equilibrium means: for every $\epsilon > 0$, *almost all* the vehicles are within ϵ-neighborhood of their corresponding equilibrium positions. In a practical scenario, this may not be realistic. We, therefore, restrict ourselves entirely to the ℓ^∞-stability analysis of systems governed by (2). Such stability analysis over $(\ell^\infty, \|\cdot\|_\infty)$ space falls under the general setting of stability analysis over an infinite dimensional Banach space, which is a recent topic of interest (see [3,4]). In this paper we provide elegant

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necessary and sufficient conditions for the ℓ^∞-stability of systems governed by (2) in terms of spectral radius of $A(e^{i\omega}, e^{-i\omega})$ and operator norm. These necessary and sufficient conditions may not always be easy to check; so, we also provide easily implementable necessary conditions and sufficient conditions for ℓ^∞-stability. These tests can be used to conclude or rule out ℓ^∞-stability.

1.1. Notation

We denote the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. We use the symbol $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{C}$ in statements that hold true for both $\mathbb{R}$ and $\mathbb{C}$. The set of integers is denoted by $\mathbb{Z}$; while the symbols $\mathbb{N}$ and $\mathbb{N}_0$ are used to denote the set of positive integers $\{1, 2, \ldots\}$ and the set of non-negative integers $\{0, 1, 2, \ldots\}$, respectively.

We use I to denote the identity operator. Transpose of a vector $\mathbf{v}$ (a matrix B) is denoted by $\mathbf{v}'$ (B'). The symbol $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$ is used to denote the space of $\mathbb{F}^n$ valued bidirectional sequences; i.e., $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) := \{\mathbf{a} : \mathbb{Z} \to \mathbb{F}^n\}$. To denote the zero element in $\mathbb{F}^n$ and $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$ we use boldface $\mathbf{0}$; and we expect it to be clear from the context. For $\mathbf{x} \in \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$, $\mathbf{x}(j)$ is used to denote the value of $\mathbf{x}$ at $j \in \mathbb{Z}$; therefore, $\mathbf{x}(j) \in \mathbb{F}^n$, $\forall j \in \mathbb{Z}$. We write $\mathbf{x}(j) = *$, when the exact value of $\mathbf{x}(j)$ is irrelevant. Analogously for $\mathbf{v} \in \mathbb{F}^n$, $\mathbf{v}(j)$ is used to denote the jth component of $\mathbf{v}$.

Laurent polynomial ring in a variable σ with coefficients from $\mathbb{F}$ is denoted as $\mathbb{F}[\sigma, \sigma^{-1}]$. We use i to denote $\sqrt{-1}$, unless specified otherwise. The unit circle, the closed unit disc and the open unit disc in $\mathbb{C}$ centered at the origin are denoted as:

$$S_{\mathbb{C}}(0, 1) := \{z \in \mathbb{C} : |z| = 1\}, \tag{3a}$$

$$B_{\mathbb{C}}(0, 1) := \{z \in \mathbb{C} : |z| \leq 1\}, \tag{3b}$$

$$B^o_{\mathbb{C}}(0, 1) := \{z \in \mathbb{C} : |z| < 1\}. \tag{3c}$$

1.2. Objective, overview and motivation

Consider the *left shift operator* $\sigma : \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) \to \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$, which is defined as $(\sigma \mathbf{x})(j) := \mathbf{x}(j+1)$. Its inverse is the right shift operator, denoted as σ^{-1}. It follows that a Laurent polynomial matrix $A(\sigma, \sigma^{-1}) = \left(\sum_{j=-m}^{p} A_j \sigma^j\right) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$, where $A_j \in \mathbb{R}^{n\times n}$ for $j \in \{-m, \ldots, p\}$, is a well defined operator on $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$; i.e., $A(\sigma, \sigma^{-1}) : \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) \to \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$. In this paper, we study the following infinite dimensional discrete autonomous system:

$$\mathbf{x}_{k+1}(\cdot) := A(\sigma, \sigma^{-1})\,\mathbf{x}_k(\cdot), \tag{4}$$

where $A(\sigma, \sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$ and $\mathbf{x}_k \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$, $\forall\, k \in \mathbb{N}_0$. The trajectories satisfying (4) can be written as:

$$\mathbf{x}_k(\cdot) := A(\sigma, \sigma^{-1})^k\,\mathbf{x}_0(\cdot), \tag{5}$$

where $\mathbf{x}_0 \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ is an initial condition.

Later in Section 2.2 we explain that, $A(\sigma, \sigma^{-1})$ is a continuous linear operator on $\ell^\infty(\mathbb{Z}, \mathbb{F}^n)$. In this paper, we obtain necessary and sufficient conditions for the ℓ^∞-stability of systems given by (4). We also give easy to check necessary conditions and sufficient conditions for the ℓ^∞-stability of such systems. Stability analysis of systems given by (4) is closely related to the stability analysis of discrete 2-D autonomous systems in general (see [5,6]); and particularly to the stability analysis of time relevant discrete 2-D autonomous systems (see [7]). When time relevant discrete 2-D autonomous systems are brought down to the state space form, the dynamics is exactly same as the one given in (4).

2. Mathematical preliminaries

2.1. Bounded linear operators

Here we briefly mention some preliminaries from functional analysis; reader can refer to [8–10] for a detailed treatment on these topics. We are interested in the normed subspace $(\ell^\infty(\mathbb{Z}, \mathbb{F}^n), \|\cdot\|_\infty)$ of $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$; for $\mathbf{x} \in \ell^\infty(\mathbb{Z}, \mathbb{F}^n)$,

$$\|\mathbf{x}\|_\infty := \sup\{\|\mathbf{x}(j)\|_\infty : j \in \mathbb{Z}\}. \tag{6}$$

Let $(X, \|\cdot\|_x)$ be any normed space over $\mathbb{F}$. Let T be a linear operator on a normed space X. The linear operator T is continuous if and only if there exists $\alpha > 0$ such that:

$$\|T(\mathbf{y})\|_x \leq \alpha\|\mathbf{y}\|_x, \quad \forall\, \mathbf{y} \in X. \tag{7}$$

Therefore, continuous linear operators are also called as *bounded linear operators*. The space of bounded linear (or continuous linear) operators on X is denoted as $BL(X)$; it is a normed space with the following induced operator norm: for $T \in BL(X)$,

$$\|T\|_x := \sup\{\|T(\mathbf{y})\|_x : \mathbf{y} \in X \text{ and } \|\mathbf{y}\|_x \leq 1\} \tag{8}$$

$$= \inf\{\alpha \in \mathbb{R} : \|T(\mathbf{y})\|_x \leq \alpha\|\mathbf{y}\|_x, \text{ for all } \mathbf{y} \in X\}. \tag{9}$$

The inequality,

$$\|T(\mathbf{y})\|_x \leq \|T\|_x\,\|\mathbf{y}\|_x, \quad \forall\, \mathbf{y} \in X \tag{10}$$

is called *the basic inequality*. The operator $T \in BL(X)$ is said to be invertible (in $BL(X)$), if T is bijective and the inverse map, T^{-1}, also belongs to $BL(X)$. For $T \in BL(X)$, *the eigenspectrum* $\Lambda_e(T)_X$, *the spectrum* $\Lambda(T)_X$, *the resolvent set* $\Lambda^c(T)_X$ and *the spectral radius* $\rho(T)_X$ are defined as follows:

$$\Lambda_e(T)_X := \{\lambda \in \mathbb{F} \mid (\lambda I - T) \text{ is not one-one}\}, \tag{11a}$$

$$\Lambda(T)_X := \{\lambda \in \mathbb{F} \mid (\lambda I - T) \text{ is not invertible}\}, \tag{11b}$$

$$\Lambda^c(T)_X := \mathbb{F} \setminus \Lambda(T)_X, \tag{11c}$$

$$\rho(T)_X := \max\{|\lambda| : \lambda \in \Lambda(T)_X\}. \tag{11d}$$

It follows from the definition that, $\Lambda_e(T)_X \subseteq \Lambda(T)_X$. If X is a finite dimensional vector space, then $\Lambda_e(T)_X = \Lambda(T)_X$.

2.2. Laurent polynomial matrix operator

Consider a Laurent polynomial matrix $A(\sigma, \sigma^{-1}) = \left(\sum_{j=-m}^{p} A_j \sigma^j\right) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$ in the shift operator σ. For ease of notation, we use L_A to denote the linear operator on $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$[1] corresponding to the Laurent polynomial matrix $A(\sigma, \sigma^{-1})$. Now, the trajectories satisfying (4) can also be written as:

$$\mathbf{x}_k = L_A^k\,\mathbf{x}_0, \tag{12}$$

where $\mathbf{x}_0 \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ is an initial condition.

Note that, for a given $\mathbf{x} \in \ell^\infty(\mathbb{Z}, \mathbb{F}^n)$,

$$\begin{aligned}(L_A\,\mathbf{x})(r) &= A(\sigma, \sigma^{-1})\,\mathbf{x}(r) \\ &= \sum_{j=-m}^{p} A_j\,\mathbf{x}(r+j) \\ &= \begin{bmatrix} A_{(-m)} & A_{(-m+1)} & \cdots & A_0 & \cdots & A_{p-1} & A_p \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{x}(r-m) \\ \mathbf{x}(r-m+1) \\ \vdots \\ \mathbf{x}(r) \\ \vdots \\ \mathbf{x}(r+p-1) \\ \mathbf{x}(r+p) \end{bmatrix},\end{aligned} \tag{13}$$

[1] Though $A(\sigma, \sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$, later for ℓ^∞-stability analysis of the system given by (4), we consider $A(\sigma, \sigma^{-1})$ as an operator over $\mathbb{C}^\infty(\mathbb{Z}, \mathbb{C}^n)$ also.