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Continuous-time singular linear–quadratic control: Necessary and sufficient conditions for the existence of regular solutions[☆]

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ABSTRACT

The purpose of this paper is to provide a full understanding of the role that the constrained generalized continuous algebraic Riccati equation plays in singular linear–quadratic (LQ) optimal control. Indeed, in spite of the vast literature on LQ problems, only recently a sufficient condition for the existence of a non-impulsive optimal control has for the first time connected this equation with the singular LQ optimal control problem. In this paper, we establish four equivalent conditions providing a complete picture that connects the singular LQ problem with the constrained generalized continuous algebraic Riccati equation and with the geometric properties of the underlying system.

1. Introduction

This paper addresses the continuous-time linear–quadratic (LQ) optimal control problem when the matrix weighting the input in the cost function, traditionally denoted by R, is possibly singular. This problem has a long history. It has been investigated in several papers and with the use of different techniques, see [1–5] and the references cited therein. In particular, in the classical contributions [1,2] it was proved that an optimal solution of the singular LQ problem exists for all initial conditions if the class of allowable controls is extended to include distributions. In the discrete time, the solution of regular and singular finite and infinite-horizon LQ problems can be found resorting to the so-called *constrained generalized discrete algebraic Riccati equation*, see [6,7] and also [8]. A similar generalization has been carried out for the continuous-time algebraic Riccati equation in [9], where the *constrained generalized Riccati equation* was defined in such a way that the inverse of R appearing in the standard Riccati equation is replaced by its pseudo-inverse. On the other hand, until very recently this counterpart of the generalized discrete algebraic Riccati equation was only studied without any understanding of its links with the linear–quadratic optimal control problem.

The recent paper [10] was the first attempt to provide a description of the role played by the constrained generalized continuous algebraic Riccati equation in singular LQ optimal control problems. Such role does not trivially follow from the analogy with the discrete case, as one can immediately realize by considering the fact that in the continuous time, whenever the optimal control involves distributions, none of the solutions of the constrained generalized Riccati equation is optimizing. In particular, in [10] it was shown that when the continuous-time constrained generalized Riccati equation possesses a symmetric solution, the corresponding LQ problem admits a *regular* (i.e. impulse-free) solution, and an optimal control can always be expressed as a state-feedback. This is just a single trait of a rich picture where necessary and sufficient conditions for the existence of regular solutions are given in terms of the algebraic and geometric structures of the underlying system. The purpose of this paper is to provide a full illustration of this picture which nicely complements the list of possible situations discussed in the pioneering work [2, see p. 332].

Notation. The image and the kernel of matrix M are denoted by $\operatorname{im} M$ and $\ker M$, respectively; the transpose and the Moore–Penrose pseudo-inverse of M are denoted by $M^{\top}$ and $M^{\dagger}$, respectively. Given a system in state-space form, we denote by $\mathscr{V}^\star$ the corresponding largest output-nulling subspace, by $\mathscr{S}^\star$ the smallest input containing subspace, and by $\mathscr{R}^\star$ the largest reachability output-nulling subspace, see [11] for details.

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1.1. Preliminaries

Let $Q, A \in \mathbb{R}^{n\times n}$, $B, S \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{m\times m}$. We make the following standing assumption:

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} = \Pi^{\mathrm{T}} \geq 0. \tag{1}$$

Thus, the *Popov matrix* Π can be factorized in terms of two matrices $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ as

$$\Pi = \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} [C \quad D]. \tag{2}$$

We define Σ to be the triple (A, B, Π). The classic LQ optimal control problem associated to Σ can be stated as follows.

Problem 1. Find a piecewise continuous control input $u(t),\ t \geq 0$, that minimizes the performance index

$$J_\infty(x_0, u) = \int_0^\infty [x^{\mathrm{T}}(t) \quad u^{\mathrm{T}}(t)] \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{3}$$

subject to the constraint

$$\dot{x}(t) = A\,x(t) + B\,u(t), \qquad x(0) = x_0 \in \mathbb{R}^n. \tag{4}$$

We consider u to be a solution of Problem 1 only if the corresponding value of the performance index is finite.[1]

It is well-known that when R is positive definite, an optimal control exists (and is indeed unique) if and only if there exists a control input for which the performance index J_∞ is finite. This is a very mild condition that admits an elegant characterization in terms of the system matrices (see Remark 1). If R is only positive semidefinite, in general Problem 1 does not admit solutions. In fact, to guarantee existence, we need to consider a relaxed problem where the control input can contain distributions (Dirac delta distributions and its derivatives). To see this fact, consider the simple case where $n = m = 1$, $A = S = R = 0$, $Q = B = 1$. In this case, the feedback control $u_k(t) = -k\,x(t),\ k \geq 0$, generates the performance index $J_\infty(x_0, u_k) = \frac{x_0^2}{2k}$. Clearly, for any given $x_0, J_\infty(x_0, u_k)$ can be made arbitrarily close to 0 by suitably choosing the constant k to be sufficiently large. In this case 0 is not the minimum but only the infimum of the values of the performance index as the control input $u(t)$ varies among piecewise continuous functions. On the other hand if we are allowed to resort to distributional control input, it is easy to see that the infimum is indeed a minimum as it can be attained by taking $u(t) = -x_0\delta(t)$, with $\delta(t)$ being the Dirac delta distribution.

We shall investigate the conditions under which Problem 1 admits solutions (which are, by definition, non-impulsive) in the general case where R is allowed to be singular. To this end a key role will be played by the following matrix equation

$$X\,A + A^{\mathrm{T}}\,X - (S + X\,B)\,R^{\dagger}\,(S^{\mathrm{T}} + B^{\mathrm{T}}X) + Q = 0. \tag{5}$$

Eq. (5) is often referred to as the *generalized continuous algebraic Riccati equation* GCARE(Σ), and represents a generalization of the classic continuous algebraic Riccati equation CARE(Σ)

$$X\,A + A^{\mathrm{T}}\,X - (S + X\,B)\,R^{-1}\,(S^{\mathrm{T}} + B^{\mathrm{T}}X) + Q = 0, \tag{6}$$

arising in infinite-horizon LQ problems since in the present setting R is allowed to be singular. Eq. (5) along with the additional condition

$$\ker R \subseteq \ker(S + X\,B), \tag{7}$$

is usually referred to as *constrained generalized continuous algebraic Riccati equation*, and is denoted by CGCARE(Σ). Observe that from (1) we have $\ker R \subseteq \ker S$, which implies that (7) is equivalent to $\ker R \subseteq \ker(X\,B)$.

The following notation is used throughout the paper. We denote by $G \stackrel{\text{def}}{=} I_m - R^{\dagger}R$ the orthogonal projector that projects onto $\ker R$. Moreover, we consider a non-singular matrix $T = [T_1 \;\;|\;\; T_2]$where $\mathrm{im} T_1 = \mathrm{im} R$ and $\mathrm{im} T_2 = \mathrm{im} G$, and we define $B_1 \stackrel{\text{def}}{=} B\,T_1$ and $B_2 \stackrel{\text{def}}{=} B\,T_2$. Finally, to any $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ we associate the matrices

$$Q_X \stackrel{\text{def}}{=} Q + A^{\mathrm{T}}X + X\,A, \qquad S_X \stackrel{\text{def}}{=} S + X\,B, \tag{8}$$

$$K_X \stackrel{\text{def}}{=} R^{\dagger}\,(S^{\mathrm{T}} + B^{\mathrm{T}}\,X) = R^{\dagger}\,S_X^{\mathrm{T}}, \qquad A_X \stackrel{\text{def}}{=} A - B\,K_X, \tag{9}$$

$$\Pi_X \stackrel{\text{def}}{=} \begin{bmatrix} Q_X & S_X \\ S_X^{\mathrm{T}} & R \end{bmatrix}. \tag{10}$$

The CGCARE(Σ) is strictly connected to the LMI

$$\Pi_X \geq 0. \tag{11}$$

Indeed, by taking the generalized Schur complement of R in Π_X, is easy to see that (11) is equivalent to the constrained generalized continuous algebraic Riccati inequality CGCARI(Σ)

$$\begin{aligned} &X\,A + A^{\mathrm{T}}\,X - (S + X\,B)\,R^{\dagger}\,(S^{\mathrm{T}} + B^{\mathrm{T}}X) + Q \geq 0, \\ &\ker R \subseteq \ker(S + X\,B) \end{aligned} \tag{12}$$

and the symmetric solutions of CGCARE(Σ) are indeed the solutions of LMI (11) for which the rank of Π_X is minimum.

2. Main result

The main result of this paper is the following theorem, whose proof will be developed in several steps in the sequel.

Theorem 1. *The following statements are equivalent:*

(A) *For every $x_0 \in \mathbb{R}^n$, Problem 1 has a solution;*
(B) *There exists a symmetric and positive semidefinite solution of CGCARE(Σ);*
(C) *There exists a symmetric solution of CGCARE(Σ), and for each $x_0 \in \mathbb{R}^n$, there exists $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite;*
(D) *For any factorization (2), the subspaces $\mathscr{S}^\star$ and $\mathscr{R}^\star$ of the quadruple (A, B, C, D) coincide, and for each initial state $x_0 \in \mathbb{R}^n$, there exists $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite.*

If any of these conditions holds an optimal solution can be obtained by static state feedback and is therefore in $\mathscr{C}_\infty[0, \infty)$.

Remark 1. Existence, for each x_0, of a control function $u_0(t)$ such that $J_\infty(x_0, u_0)$ is finite is a very natural condition. Its testability, however, is not obvious. It has been shown in [12] that such condition is equivalent to the following neat and easily testable geometric condition:

$$\mathscr{V}^\star + \mathscr{R}(A, B) + \mathscr{X}_{\mathrm{stab}} = \mathbb{R}^n,$$

where $\mathscr{V}^\star$ is the largest output-nulling subspace of the quadruple (A, B, C, D), $\mathscr{R}(A, B)$ is the reachable subspace (i.e., the smallest A-invariant subspace containing the range of B), and $\mathscr{X}_{\mathrm{stab}}$ is the A-invariant subspace corresponding to the asymptotically stable uncontrollable eigenvalues of A (so that, in other words, the sum $\mathscr{R}(A, B) + \mathscr{X}_{\mathrm{stab}}$ is the stabilizable subspace of the pair (A, B)).

[1] We make this remark since, if the cost is unbounded for every control, one might alternatively say that all controls are optimal since they all lead to the same value of the performance index.