Systems & Control Letters 93 (2016) 50–57

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Exponential stability for formation control systems with generalized controllers: A unified approach

Zhiyong Sun [a,*], Shaoshuai Mou [b], Brian D.O. Anderson [a], Ming Cao [c]

[a] National ICT Australia and Research School of Engineering, Australian National University, Canberra, Australia
[b] School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN, USA
[c] Faculty of Mathematics and Natural Sciences, ENTEG, University of Groningen, The Netherlands

ARTICLE INFO

Article history:
Received 25 November 2015
Accepted 24 February 2016
Available online 13 April 2016

Keywords:
Exponential convergence
Graph rigidity
Distance-based formations

ABSTRACT

This paper discusses generalized controllers for distance-based rigid formation shape stabilization and aims to provide a unified approach for the convergence analysis. We consider two types of formation control systems according to different characterizations of target formations: minimally rigid target formation and non-minimally rigid target formation. For the former case, we firstly prove the local *exponential stability* for rigid formation systems when using a general form of shape controllers with certain properties. From this viewpoint, different formation controllers proposed in previous literature can be included in a unified framework. We then extend the result to the case that the target formation is non-minimally rigid, and show that exponential stability of the formation system is still guaranteed with generalized controllers.

1. Introduction

Formation control of networked multi-agent systems has received considerable attention in recent years due to its extensive applications. One problem of extensive interest is *formation shape control*, i.e. on designing controllers to achieve or maintain a geometrical shape for the formation Oh et al. [1]. As reviewed in [1], the existing results on formation control can be characterized into position-, displacement-, and distance-based control strategies according to different types of sensed and controlled variables. Among these formation control strategies, distance-based formation control receives particular interest as it allows reduced requirement on the sensing capability for individual agent compared to the other two strategies. Thus, this paper focuses on distance-based formation control. In particular, we confine our attention in this paper to *undirected* rigid formations, while relevant discussions on directed formation control can be found in e.g. [2].

With underpinnings from rigid graph theory, it has been established that the formation shape can be achieved by controlling a certain set of inter-agent distances Olfati-Saber and Murray [3]; Anderson et al. [4]; Krick et al. [5]. In the rigid formation stabilization problem, one typical controller that has been studied extensively in the literature takes the following form (see e.g. [5]):

$$\dot{p}_i = \sum_{j\in\mathcal{N}_i}(p_j - p_i)(\|p_j - p_i\|^2 - d_{ij}^2) \tag{1}$$

where the definitions underpinning the notation will become clear in later sections. The above controller (1), which is derived from a well-defined potential function, serves as a standard control law for stabilizing rigid formations. The dynamics of the formation control system (1) have been investigated in several succeeding papers, e.g. [6–8]. We mention that alternative kinds of formation controllers other than the one in (1) are also available, which have been reported sparsely in the literature (see e.g. [9–12]).

The main objective of this paper is to analyze general forms of formation controllers to stabilize rigid shapes. The main contributions of this paper include a unified approach to discuss the convergence and controller performance of generalized formation controllers, and the associated *exponential stability* of general formation systems when certain properties of the potential function are satisfied. We show that for a large family of formation control systems which generalizes most existing formation controllers in the literature, the exponential stability of the distance error system can be guaranteed (for a list of such controllers, see Section 3.1). As is well-known in the control field, exponential stability has the robust property against small perturbations Khalil [13]. Such robust property has been employed in recent papers Mou et al. [14]; Sun et al. [15]; Garcia de Marina

* Corresponding author.
E-mail address: zhiyong.sun@anu.edu.au (Z. Sun).

http://dx.doi.org/10.1016/j.sysconle.2016.02.022
0167-6911/

et al. [16] to study the behavior of rigid undirected formations when there are mismatches or discrepancies between neighbor agents' distance measurements. Exponential stability enables that the desired equilibrium of the derived distance error system under controller (1) is still exponentially stable with small distance perturbations. Note that in the analysis of the formation robustness issue in [14–16], the formation control system is limited to the case of the controller in (1). To this end, this paper shows that formation distance error systems with generalized controllers, which include many special forms of formation controllers studied in the literature, also inherit this robustness property as a consequence of the exponential stability.

In this paper, we first consider formation stabilization control when the target formation is minimally rigid, which is a common assumption in the literature. We list several requirements of the potential and gradient functions associated with the distributed control for each agent which render an exponential convergence of the formation system. We give an explicit lower bound of the convergence rate, and also discuss several properties of the generalized formation control systems. By deriving a reduced distance error system via the decomposition principle of a rigid framework, we then extend the analysis from the minimally rigid case to the non-minimally rigid case (motivated by the analysis in [14,17]), and further prove that the exponential convergence still holds for non-minimally rigid formation control when generalized formation controllers are applied.

A preliminary version of this paper has been presented in [18]. The extensions of this paper compared to Sun et al. [18] include complete proofs for all the key results which were omitted in [18], detailed discussions on certain properties of generalized formation controllers, new simulation results, and especially, a new section to discuss convergence property for non-minimally rigid formations.

The rest of this paper is organized as follows. In Section 2, preliminary concepts on graph theory and rigidity theory are introduced. In Section 3, we provide detailed analysis on generalized controllers and prove the exponential stability property for minimally rigid formations. Section 4 discusses exponential convergence for formation control systems when the target formation is non-minimally rigid. In Section 5, two sets of simulation examples are provided to demonstrate the controller performance. Finally, Section 6 concludes this paper.

2. Basic concepts on graph and rigidity theory

Consider an undirected graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ij}\} \in \mathbb{R}^{m\times n}$, whose entries are defined as (with arbitrary edge orientations for the *undirected* formations considered here)

$$h_{ij} = \begin{cases} 1, & \text{the } i\text{th edge sinks at node } j \\ -1, & \text{the } i\text{th edge leaves node } j \\ 0, & \text{otherwise.} \end{cases}$$

For a connected undirected graph, one has $\ker(H) = span\{\mathbf{1}_n\}$. Note that for a rigid formation modeled by an *undirected* graph considered in this paper, the orientation of each edge for writing the incidence matrix can be defined arbitrarily and the stability analysis in the next sections remains unchanged.

Let $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$ denote a point that is assigned to $i \in \mathcal{V}$. The stacked vector $p = [p_1^T, p_2^T, \ldots, p_n^T]^T \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^d$. By introducing the matrix $\bar{H} := H \otimes I_d \in \mathbb{R}^{dm\times dn}$, one can construct the relative position vector as an image of $\bar{H}$ from the position vector p:

$$z = \bar{H}p \tag{2}$$

where $z = [z_1^T, z_2^T, \ldots, z_m^T]^T \in \mathbb{R}^{dm}$, with $z_k \in \mathbb{R}^d$ being the relative position vector for the vertex pair defined by the kth edge.

Using the same ordering of the edge set $\mathcal{E}$ as in the definition of H, the rigidity function $r_{\mathcal{G}}(p) : \mathbb{R}^{dn} \to \mathbb{R}^m$ associated with the framework $(\mathcal{G}, p)$ is given as:

$$r_{\mathcal{G}}(p) = \frac{1}{2}\left[\cdots, \|p_i - p_j\|^2, \ldots\right]^T, \quad (i, j) \in \mathcal{E} \tag{3}$$

where the norm is the standard Euclidean norm, and the kth component in $r_{\mathcal{G}}(p)$, $\|p_i - p_j\|^2$, corresponds to the squared length of the relative position vector z_k which connects the vertices i and j.

The rigidity of frameworks is then defined as follows.

Definition 1 (*Asimow and Roth [19]*)**.** A framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^d$ if there exists a neighborhood $\mathbb{U}$ of p such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap \mathbb{U} = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)) \cap \mathbb{U}$ where $\mathcal{K}$ is the complete graph with the same vertices as $\mathcal{G}$.

In the following, the set of all frameworks $(\mathcal{G}, p)$ which satisfy the distance constraints is referred to as the set of *target formations*. Let $(d_{k_{ij}})$ denote the desired distance of edge k in the target formation which links agent i and j. We further define

$$e_{k_{ij}} = \|p_i - p_j\|^2 - (d_{k_{ij}})^2 \tag{4}$$

to denote the squared distance error for edge k. Note we may use e_k and d_k occasionally for notational convenience if no confusion is expected. Also in the context of formation control under discussions, the term *framework* will be occasionally referred to as *formation* realized by a set of n agents. Thus we may also use the two terms, *framework* and *formation* interchangeably if no confusion is caused. Define the distance error vector as $e = [e_1, e_2, \ldots, e_m]^T$.

One useful tool to characterize the rigidity property of a framework is the rigidity matrix $R \in \mathbb{R}^{m\times dn}$, which is defined as

$$R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p}. \tag{5}$$

It is not difficult to see that the row of the rigidity matrix R corresponding to $\{(i, j) \in \mathcal{E}\}$ takes the following form

$$[\mathbf{0}_{1\times d}, \ldots, (p_i - p_j)^T, \ldots, \mathbf{0}_{1\times d}, \ldots, (p_j - p_i)^T, \ldots, \mathbf{0}_{1\times d}]. \tag{6}$$

Each edge gives rise to a row of R, and, if an edge links vertices i and j, then the nonzero entries of the corresponding row of R are in the columns from $di - (d - 1)$ to di and from $dj - (d - 1)$ to dj.

Eq. (2) shows that the relative position vector lies in the image of $\bar{H}$. Thus one can redefine the rigidity function, $g_{\mathcal{G}}(z) : Im(\bar{H}) \to \mathbb{R}^m$ as $g_{\mathcal{G}}(z) = \frac{1}{2}\left[\|z_1\|^2, \|z_2\|^2, \ldots, \|z_m\|^2\right]^T$. From (2) and (5), one can obtain the following simple form for the rigidity matrix

$$R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p} = \frac{\partial g_{\mathcal{G}}(z)}{\partial z}\frac{\partial z}{\partial p} = Z^T\bar{H} \tag{7}$$

where $Z = \text{diag}\{z_1, z_2, \ldots, z_m\}$. It is obvious that the entries of $R(p)$ are also functions of z, and we will also write it as $R(z)$. The rigidity matrix will be used to determine the infinitesimal rigidity of the framework, as shown in the following theorem.

Theorem 1 (*Hendrickson [20]*)**.** *Consider a framework $(\mathcal{G}, p)$ in d-dimensional space with $n \geq d$ vertices and m edges. It is infinitesimally rigid if and only if*

$$rank(R(p)) = dn - d(d + 1)/2. \tag{8}$$