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Invariance and monotonicity of nonlinear systems on time scales

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ABSTRACT

Invariance of a subset of the state space with respect to a control system on an arbitrary time scale is studied. This includes in particular continuous-time and discrete-time systems. The invariance is characterized with the aid of natural operators associated to the system: shifts and skew derivations. For different time scales the criteria take different forms. Then for polynomial systems algebraic criteria of invariance based on Positivstellensatz from real algebraic geometry are developed. A relation between invariance and monotonicity is exhibited and the results on invariance are used to characterize monotonicity.

1. Introduction

Invariance of a subset of the state space with respect to the dynamics of a control system is a crucial property in many areas of control theory and practice. For example, a system is positive, if the positive cone of the state space is preserved by the system's dynamics. It is customary to extend the notion of positivity to systems that preserve some other cone with good properties, not necessarily the positive cone. There is also a simple relation between monotonicity of the system and invariance. The system is monotone if its dynamics preserves a partial order defined on the state space. The partial order is often introduced with the aid of a cone. Angeli and Sontag [1] preferred to define the order in a more general way, replacing a cone by an arbitrary subset of the product of the state space with itself that yields a partial order. We go one step further, following a suggestion in [1], and consider monotonicity with respect to a relation that is not necessarily a partial order. We study monotonicity only with respect to states and not with respect to states and controls as is done in [1]. Thus we are closer to the original concepts of Smith [2], where only dynamical systems without control were studied. This is caused by our choice of not imposing any structure on the set of control values, as was done e.g. in our studies of positive realizations [3].

The goal of the paper is to find nice characterizations of invariance and monotonicity for both continuous- and discrete-time systems. The language of calculus on time scales allows for doing this simultaneously for the two classes. Moreover, we are able to accommodate systems on non-uniform discrete time domains, studied e.g. in [4,5], and systems with a mixed time. The theory of control systems on time scales is quite developed now, including linear and nonlinear systems (see e.g. [6,7]). Similar study concerning monotonicity of systems on time scales has been done in [8], but with a more traditional concept of partial order based on cones, which have also been used to express the criteria of invariance. The common feature of [8] and this paper is that the nature of time, discrete or continuous, strongly influences the criterion of invariance, so in the general case we must have two conditions, of which only one is triggered for a particular instance of time. We express these conditions using two fundamental operators acting on real functions defined on the state space: the skew derivation and the shift. They were used before in [3] for formulating the conditions of positive realizability. When we choose the case of continuous-time systems, we get a known characterization, but expressed in a different language. The usual formulation relies on the notion of tangent cone [1]. The discrete-time case is more straightforward. But since the discrete-time system is described by a delta differential equation and not by the usual shift equation, the characterization of invariance or monotonicity takes a different form.

In the last section we retreat to a more structured class of polynomial systems. Such a system is described by a family of polynomial maps (vector fields) parameterized by control values, and the set whose invariance is examined is defined by polynomial inequalities, i.e. it is semialgebraic. This allows to use well developed machinery of real algebraic geometry (see e.g. [9]). In particular we use a characterization of functions that are nonnegative on a semialgebraic set, known as *Positivstellensatz* [10,11]. Such characterization has been extended to real analytic geometry [12], so

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one can try to generalize the algebraic criteria of invariance and monotonicity for polynomial systems presented in this paper to the real analytic case. The language and results of real analytic geometry, especially real analytic *Nullstellensatz*, were used in [13,14] to develop algebraic criteria of local observability. The results of this section are new even for continuous-time or discrete-time systems. However one has to remember that they are restricted to the polynomial case.

Monotone systems are used as models of many concrete systems that appear in various areas of science and technology. Examples of such systems can be found e.g. in [2]. Two important classes of competitive and cooperative systems are common in economics, social sciences and biology. They also appear in robotics to model swarms of robots [15].

2. Preliminaries

By $\mathbb{R}$ we shall denote the set of all real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers (without 0). $\mathbb{R}_+$ will mean the set of nonnegative real numbers, $\mathbb{Z}_+$ the set of nonnegative integers, i.e. $\mathbb{N} \cup \{0\}$, and $\mathbb{R}^k_+$ the set of all column vectors in $\mathbb{R}^k$ with nonnegative components.

By $x^i : \mathbb{R}^n \to \mathbb{R}$ we denote the coordinate function: $x^i(x) = x_i$ for $x = (x_1, \ldots, x_n)$. By $C^1(\mathbb{R}^n)$ we shall denote the algebra of all real-valued functions defined on $\mathbb{R}^n$ with continuous partial derivatives. If $\varphi \in C^1(\mathbb{R}^n)$, then we say that φ is of class C^1. The same concerns a map $g : \mathbb{R}^n \to \mathbb{R}^k$ whose components belong to $C^1(\mathbb{R}^n)$. The algebra of real polynomial functions on $\mathbb{R}^n$ will be denoted by $\mathbb{R}[x_1, \ldots, x_n]$.

Calculus on time scales is a generalization of the standard differential calculus and the calculus of finite differences. For convenience of the reader we recall here the basic definitions and facts used in this paper. More information can be found e.g. in [16].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. In particular $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = a\mathbb{Z}$ for $a > 0$ and $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$ are time scales. In the whole paper we consider on $\mathbb{T}$ the relative topology induced from $\mathbb{R}$. If $t_0, t_1 \in \mathbb{R}$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the ordinary closed interval with $\mathbb{T}$. Similar notation is used for open, half-open or infinite intervals.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$ when $\sup \mathbb{T}$ is finite; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) := \inf \mathbb{T}$ when $\inf \mathbb{T}$ is finite; the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

In the whole paper we will assume that $\mathbb{T}$ is *forward infinite*, i.e. for every $t \in \mathbb{T}$ there are infinitely many points in $\mathbb{T}$ that are greater than t.

Let $\gamma : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. The *delta derivative of* γ *at* t, denoted by $\gamma^{\Delta}(t)$, is the real number with the property that given any $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ such that

$$|(\gamma(\sigma(t)) - \gamma(s)) - \gamma^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. If $\gamma^{\Delta}(t)$ exists, then we say that γ *is delta differentiable at* t. Moreover, we say that γ is *delta differentiable* on $\mathbb{T}$, provided $\gamma^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $\gamma : \mathbb{T} \to \mathbb{R}^n$ is *delta differentiable* if all its components γ_i are delta differentiable. Then $\gamma^{\Delta}(t) := (\gamma_1^{\Delta}(t), \ldots, \gamma_n^{\Delta}(t))^T$.

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then $\gamma^{\Delta}(t) = \dot{\gamma}(t)$—the ordinary derivative. If $\mathbb{T} = a\mathbb{Z}$, then $\gamma^{\Delta}(t) = \frac{\gamma(t+a) - \gamma(t)}{a}$. If $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, then $\gamma^{\Delta}(t) = \frac{\gamma(qt) - \gamma(t)}{(q-1)t}$.

For a function $\gamma : \mathbb{T} \to \mathbb{R}$ let $\gamma^{\sigma} := \gamma \circ \sigma$.

Proposition 2.2. *If* $\gamma : \mathbb{T} \to \mathbb{R}$ *is delta differentiable, then* $\gamma^{\sigma} = \gamma + \mu\gamma^{\Delta}$.

Here is the chain rule on time scales.

Proposition 2.3. *Let* $n \in \mathbb{N}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be of class* C^1 *and* $\gamma : \mathbb{T} \to \mathbb{R}^n$ *be delta differentiable on* $\mathbb{T}$. *Then*

$$(\varphi \circ \gamma)^{\Delta}(t) = \int_0^1 \varphi'(\gamma(t) + s\mu(t)\gamma^{\Delta}(t))ds\, \gamma^{\Delta}(t),$$

where $\varphi' := (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n})$ *is the gradient of* φ.

Corollary 2.4. *For delta differentiable functions* γ_1 *and* γ_2

$$\begin{aligned}(\gamma_1\gamma_2)^{\Delta} &= \gamma_1^{\sigma}\gamma_2^{\Delta} + \gamma_1^{\Delta}\gamma_2 = \gamma_1\gamma_2^{\Delta} + \gamma_1^{\Delta}\gamma_2^{\sigma} \\ &= \gamma_1\gamma_2^{\Delta} + \gamma_1^{\Delta}\gamma_2 + \mu\gamma_1^{\Delta}\gamma_2^{\Delta}.\end{aligned}$$

Let now $H : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$. Consider the delta differential equation

$$x^{\Delta}(t) = H(x(t), t). \tag{1}$$

A *solution* to (1) is a function x defined on some interval $[a, b)_{\mathbb{T}} \subseteq \mathbb{T}$ and satisfying (1). If H is continuous and is of class C^1 with respect to x (the first variable), then for every initial condition $x(t_0) = x_0$ there exists a unique forward solution defined on some interval $[t_0, t_1)_{\mathbb{T}}$, where $t_1 \in \mathbb{T}$ and $t_1 > t_0$ [16].

Now we recall the main tools that have been used to study positive realizations in [3,17]: skew derivations of algebras. They will be also useful in our study of forward invariance and monotonicity.

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and let $t \in \mathbb{T}$. As in [3] let us define $\Gamma_h^t : C^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$ by

$$\Gamma_h^t(\varphi)(x) := \int_0^1 \varphi'(x + s\mu(t)h(x))ds\, h(x), \tag{2}$$

where φ' is the gradient of φ.

Remark 2.5. If $\mu(t) > 0$ then

$$\Gamma_h^t(\varphi)(x) = \frac{1}{\mu(t)}(\varphi(x + \mu(t)h(x)) - \varphi(x)).$$

For $\mu(t) = 0$ we obtain $\Gamma_h^t(\varphi)(x) = \varphi'(x)h(x)$, so $\Gamma_h^t(\varphi)$ is then equal to $L_h(\varphi)$ – the Lie derivative of the function φ with respect to the vector field h.

Example 2.6. Let $\varphi = x^i$. Then for any time scale $\mathbb{T}$ and for any $t \in \mathbb{T}$

$$\Gamma_h^t(\varphi)(x) = h_i(x).$$

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and let $t \in \mathbb{T}$. Define $\sigma_h^t : C^1(\mathbb{R}^n) \to C^1(\mathbb{R}^n)$ by

$$\sigma_h^t(\varphi)(x) := \varphi(x + \mu(t)h(x)).$$

We will call it a *shift*.

Observe that the map σ_h^t is a homomorphism of the ring $C^1(\mathbb{R}^n)$. For $\mu(t) = 0$ this map is the identity map.

We have natural relations between Γ_h^t and σ_h^t.

Proposition 2.7 *([3])*.

1. *For* $\varphi, \psi \in C^1(\mathbb{R}^n)$, $\Gamma_h^t(\varphi\psi) = \Gamma_h^t(\varphi)\psi + \sigma_h^t(\varphi)\Gamma_h^t(\psi) = \Gamma_h^t(\varphi)\sigma_h^t(\psi) + \varphi\Gamma_h^t(\psi)$.
2. *For* $\varphi \in C^1(\mathbb{R}^n)$, $\sigma_h^t(\varphi) = \varphi + \mu(t)\Gamma_h^t(\varphi)$.