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Recurrence and LaSalle invariance principle

Boyang Ding [a], Changming Ding [b,*]

[a] School of Economics, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, PR China
[b] School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, PR China

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ABSTRACT

In this paper, we consider three different notions of recurrence, chain recurrence, strong chain recurrence and generalized recurrence, where the generalized recurrence is introduced by Auslander (1964) and the strong chain recurrence is derived from the idea of a strong chain defined by Easton (1978). We prove that for a quasi-gradient flow in a compact metric space, these three recurrences are equivalent. Finally, we establish an invariance principle, i.e., the ω-limit set of a precompact orbit is contained in the generalized recurrent component of 'zero derivative' set of a Lyapunov function.

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1. Introduction

The ultimate aim of one studying a dynamical system from a topological point of view is to understand its phase portrait or orbit structure. Frequently, the first step is to describe the eventual behavior of the orbits, that is to determine the limit sets. LaSalle invariance principle is one of the most important tools for convergence analysis, see [1]. In the literature, there are numberless extensions of this elegant principle for variant applications to dynamical systems in applied sciences such as electrical and mechanical engineering, biology, and physics. For instance, it was extended to infinite dimensional systems [2,3], non-autonomous differential equations [4,5] and differential inclusions [6,7]. By the invariance principle, system stability can be investigated by employing a proper Lyapunov function with semi-negative time derivative along orbits, which implies that the ω-limit set of bounded orbits is a compact subset of the largest invariant set in the 'zero derivative' set.

Of course, the largest invariant set contained in the 'zero derivative' set of a Lyapunov function may be large, thus it is hard to locate an ω-limit set. It is a challenging problem to determine an ω-limit set in the 'zero derivative' set more accurately. An excellent result in this direction was concluded by Helmke in [8], i.e., an ω-limit set is contained in an internally chain transitive component in the largest invariant set. However, in some cases, a chain transitive component is still too large. For example, consider the flow in the plane defined by $\dot{x_1} = x_1^2 + x_2^2 - 1, \dot{x_2} = 0$, which was presented by Nitecki in [9]. Clearly, the chain recurrent set (the unit disc) has a positive measure and the generalized (Auslander) recurrent set (the unit circle) has measure zero. Since a limit set is generalized recurrent, it is natural to determine the ω-limit set in a component of the generalized recurrent set, which is one of our main goals in this paper.

A point exhibits some sort of recurrent behavior when the dynamical system returns the point to itself, or to a neighborhood of itself, in a particular way. Chain recurrence is one type of recurrence with errors allowed along the orbit, which was introduced by Conley in [10]. To give a criterion of Lipschitz ergodicity, Easton [11] introduced the concept of a strong chain, which demands a 'sum error' along the orbit. In a natural way, strong chains lead to a corresponding notion of strong chain recurrence. In this paper, we deal with chain recurrence, strong chain recurrence and generalized recurrence. For the elementary properties of these recurrent motions, we refer to [10–15].

This paper is organized as follows. In the next section, we show that if a dynamical system defined in a compact metric space is quasi-gradient, i.e., the minimal sets are isolated, then chain recurrence is equivalent to strong chain recurrence. In Section 3, we first recall the generalized recurrence established by Auslander in [13]. If a point is generalized recurrent, then it is strong chain recurrent. Further, we show that for a quasi-gradient flow, generalized recurrence is equivalent to chain recurrence, although these two notions seem completely different by their definitions. Finally, in Section 4, we use the generalized recurrence to present

* Corresponding author.
E-mail address: cding@xmu.edu.cn (C. Ding).

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a strong LaSalle invariance principle. We prove that the ω-limit set of a precompact orbit is contained in a generalized recurrent component in the 'zero derivative' set of a Lyapunov function.

2. Chain recurrence

In this section, X denotes a compact metric space with metric d. For a set $A \subset X$, $\overline{A}$ denotes the closure of A. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ be the open ball with center x and radius $r > 0$. If $A \subset X$ and $r > 0$, the r-neighborhood of A is denoted by $N(A, r) = \{z \in X : d(z, A) < r\}$, where $d(z, A) = \inf\{d(z, p) : p \in A\}$. Let $\mathbb{R}$ be the real line, and $\mathbb{R}^+$ the subset of $\mathbb{R}$ consisting of nonnegative real numbers.

A dynamical system or continuous flow on X is a continuous map $\pi : X \times \mathbb{R} \to X$ satisfying (i) $\pi(x, 0) = x$ for all $x \in X$ and (ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{R}$. For brevity, we write $x \cdot t$ in place of $\pi(x, t)$, then axiom (ii) reads $(x \cdot t) \cdot s = x \cdot (t + s)$. Similarly, let $A \cdot J = \{x \cdot t : x \in A, t \in J\}$ for $A \subset X$ and $J \subset \mathbb{R}$, also write $x \cdot J$ for $\{x\} \cdot J$. In particular, if $x \in X$, the orbit of x is $\gamma(x) = x \cdot \mathbb{R}$ and the positive semi-orbits of x is $\gamma^+(x) = x \cdot \mathbb{R}^+$. A set Y in X is invariant if $\gamma(x) \subset Y$ for all $x \in Y$. A set $M \subset X$ is called minimal, if it is nonempty, closed, invariant and no proper subset of M has these properties.

The ω-limit set $\omega(x)$ of a point $x \in X$ is defined to be the set $\bigcap_{t\ge 0} \overline{x \cdot [t, +\infty)}$, equivalently $y \in \omega(x)$ means that there is a sequence $\{t_i\}$ with $t_i \to +\infty$ such that $x \cdot t_i \to y$ as $i \to +\infty$. Similarly, the α-limit set $\alpha(x)$ of $x \in X$ is the set $\bigcap_{t\le 0} \overline{x \cdot (-\infty, t]}$. The first prolongational set and first prolongational limit set are defined, respectively, by $D_1(x) = \{y \in X$: there are two sequences $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbb{R}^+$ such that $x_n \to x$ and $x_n \cdot t_n \to y\}$ and $J_1(x) = \{y \in X$: there are two sequences $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbb{R}^+$ such that $x_n \to x$, $t_n \to +\infty$ and $x_n \cdot t_n \to y\}$. Note that $D_1(x) = \gamma^+(x) \cup J_1(x)$ holds.

The notion of chain recurrence, introduced by Conley [10], is a way of getting at the recurrence properties of a dynamical system. It has remarkable connections to the structure of attractors. For two points x, y in X and two positive numbers $\epsilon, t > 0$, an (ϵ, t)-chain from x to y means a collection $\{x = x_1, x_2, \ldots, x_{n+1} = y : t_1, t_2, \ldots, t_n\}$ such that $t_i \ge t$ and $d(x_i \cdot t_i, x_{i+1}) < \epsilon$ for $1 \le i \le n$. A point $x \in X$ is *chain recurrent* if for all $\epsilon, t > 0$ there exists an (ϵ, t)-chain from x to itself. The *chain recurrent set* $\mathcal{C}$ in X is the set of all chain recurrent points. Note that $\mathcal{C}$ is closed and invariant. An invariant closed set $Y \subset X$ is *chain transitive* if for all $x, y \in Y$ and all $\epsilon, t > 0$ there exists an (ϵ, t)-chain from x to y. Clearly, if an invariant closed set Y is chain transitive, then each point in Y is chain recurrent, i.e., $Y \subset \mathcal{C}$. Also, every component of the chain recurrent set $\mathcal{C}$ is chain transitive.

Since X is compact, the chain recurrent component has the *restriction property*, i.e., (ϵ, t)-chains connecting points in $\mathcal{C}$ can be chosen to lie in $\mathcal{C}$, see [14, Theorem 3.6D] or [16, p. 429].

Definition 2.1. A chain transitive set D is reducible if there exists a nonempty invariant closed proper subset of D that is also chain transitive. A chain transitive set is said to be irreducible if it is not reducible.

Clearly, rest points and periodic orbits are irreducible chain transitive sets. Consider the differential equation $\dot{\theta} = \sin^2\theta$ in the angular coordinate on $\mathbb{S}^1$ (the unit circle in the plane $\mathbb{R}^2$), it is easy to see that $\mathbb{S}^1$ is a reducible chain transitive set. By the definition, any two different irreducible chain transitive sets are disjoint.

Property 2.2. *A subset D of X is an irreducible chain transitive set if and only if D is a minimal set.*

Proof. Let D be an irreducible chain transitive set. For an $x \in D$, if $\omega(x) \neq D$, then $\omega(x)$ is a nonempty invariant closed proper subset of D, which is also chain transitive. It is contradictory to that D is irreducible. Hence, we have $\omega(x) = D$ for any $x \in D$, it means that D is minimal. Conversely, since a minimal set is chain transitive and has no nonempty invariant closed proper subsets, it is an irreducible chain transitive set. □

To give a criterion of Lipschitz ergodicity, Easton [11] introduced the concept of a strong chain. For two points x, y in X and two positive numbers ϵ, t, the collection $\{x = x_1, x_2, \ldots, x_{n+1} = y : t_1, t_2, \ldots, t_n\}$ is said to be a strong (ϵ, t)-chain from x to y if $\sum_{i=1}^{n} d(x_i \cdot t_i, x_{i+1}) < \epsilon$ and $t_i \ge t$ for $1 \le i \le n$. A point $x \in X$ is *strong chain recurrent* if for all $\epsilon, t > 0$ there exists a strong (ϵ, t)-chain from x to x. The *strong chain recurrent set* $\mathcal{S}$ in X is the set of all strong chain recurrent points. Clearly, $\mathcal{S}$ is closed and invariant. An invariant closed set $Y \subset X$ is *strong chain transitive* if for all x, $y \in Y$ and all $\epsilon, t > 0$ there exists a strong (ϵ, t)-chain from x to y.

Note that the strong chain recurrent set does not have the restriction property. Consider the flow π corresponding to the system of differential equations $\dot{r} = r(1 - r)^3$, $\dot{\theta} = (1 - r)^2$ in polar coordinates on the unit disc $\mathbb{D}^2 = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1\}$ in the plane $\mathbb{R}^2$. Then, $\mathbb{S}^1$ is strong chain recurrent under π, however it is not strong chain recurrent under $\pi|_{\mathbb{S}^1}$.

If a point is strong chain recurrent, then it is chain recurrent, this means $\mathcal{S} \subset \mathcal{C}$. However, the converse is not true. Consider a system presented by Conley in [14] as follows. Let X denote the unit square in the plane, and the flow is generated by the vector field $\dot{x} = 0$, $\dot{y} = -xy(1 - x)(1 - y)$. It is easy to see that $\mathcal{C} = X$ and $\mathcal{S}$ is the boundary of the square. Of course, a minimal set is strong chain recurrent, but the converse is not true.

Definition 2.3. An irreducible chain transitive set D is isolated, if there exists an open neighborhood U of D such that U contains no points of other irreducible chain transitive sets. The flow π is called quasi-gradient if every irreducible chain transitive set is isolated.

If a flow π is quasi-gradient, it has some regularity, e.g., all the rest points and periodic orbits are isolated. Actually, we have the interesting result as follows.

Theorem 2.4. *If X is compact and π is quasi-gradient, then every chain recurrent point is strong chain recurrent, i.e., $\mathcal{C} = \mathcal{S}$.*

Proof. Let $x \in D \subset \mathcal{C}$, where D is a component of $\mathcal{C}$ and is chain transitive, see [14]. Recall that the chain recurrent set has the restriction property [14, Theorem 3.6D]. If we let $\pi|_D$ be the sub-flow on the connected set D, then D is also chain transitive for $\pi|_D$. Define $T = \{y \in D$: for any $\epsilon, t > 0$ there exists a strong (ϵ, t)-chain in D from x to $y\}$, which is an invariant closed set and $\omega(x) \subset T$. Since different irreducible chain transitive sets are disjoint and π is quasi-gradient, it follows from the compactness of X that there exist only a finite number of irreducible chain transitive sets for π. To see this, let $(2^X, H_d)$ be the hyperspace of X, where $2^X = \{A : A$ is a nonempty closed subset of $X\}$ and H_d is the Hausdorff metric. It is well known that 2^X is a compact metric space, see [17, Chap. 4]. If there are an infinite number of irreducible chain transitive sets $\{D_i\}$ in X, then $\{D_i\}$ has an accumulation point D^* in 2^X. The set D^* is a nonempty closed invariant set, and hence contains a minimal subset D', which is an irreducible chain transitive set by Property 2.2. Since π is quasi-gradient, there exists a neighborhood V of D' such that V contains no other points of irreducible chain transitive sets. This contradicts the fact that D^* is an accumulation point of $\{D_i\}$. Consequently, if $D \setminus T \neq \emptyset$, we take $p \in D \setminus T$ such that $d(p, T) = \delta > 0$ and there exist no any irreducible chain transitive sets in the bounded set $U = N(T, \delta) \setminus T$. Now, if there exists a sequence $\{x_n\}$ in U such that $d(x_n, T) \to 0$ and $\omega(x_n) \cap (D \setminus N(T, \delta)) \neq \emptyset$. Without loss of

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