Systems & Control Letters 87 (2016) 29–35

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Review

On the controllability of a free-boundary problem for the 1D heat equation

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ARTICLE INFO

Article history:
Received 15 January 2015
Received in revised form
24 October 2015
Accepted 27 October 2015
Available online 21 November 2015

Keywords:
Null controllability
Free-boundary problems
1D heat equation
Carleman estimates

ABSTRACT

This paper deals with the local null control of a free-boundary problem for the classical 1D heat equation with distributed controls, locally supported in space. In the main result we prove that, if the final time T is fixed and the initial state is sufficiently small, there exist controls that drive the state exactly to rest at time $t = T$.

Contents

1. Introduction. The main result

Assume that $L_0 > 0$, $T > 0$, $0 < a < b < L_* < L_0$ and $y_0 \in L^\infty(0, L_0)$ are given. In this paper, we consider free-boundary problems of the following kind:

Find $L \in C^1([0,T])$ with $L(t) > 0$ for all t and a function $y = y(x,t)$ such that

$$L(0) = L_0, \tag{1}$$

$$\begin{cases} y_t - y_{xx} = v 1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0, \quad y(L(t),t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0, L_0) \end{cases} \tag{2}$$

and, moreover,

$$y_x(L(t),t) = -L'(t), \quad t \in (0,T). \tag{3}$$

Here, we have used the notation

$$Q_L = \{ (x,t) : x \in (0, L(t)),\ t \in (0,T) \}, \quad \omega = (a,b);$$

as usual, 1_ω denotes the characteristic function of ω.

In (1)–(3), v is a control and (L, y) is an associated state. Recall that, for any $y_0 \in H_0^1(0, L_0)$ and $v \in L^2(\omega \times (0,T))$, there exists at least one *local in time* solution to (2)–(3). In other words, there

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http://dx.doi.org/10.1016/j.sysconle.2015.10.011
0167-6911/

exist $T_* \in (0, T]$ and a couple (L, y) with

$$\begin{cases} L \in C^1([0, T_*)), \quad L(t) \geq L_* \quad \text{in } [0, T_*), \\ y, y_x, y_{xx}, y_t \in L^2(Q_L^*), \\ \quad \text{with } Q_L^* := \{(x,t) : x \in (0, L(t)),\ t \in (0, T_*)\}, \\ L(0) = L_0, \quad y(x, 0) = y_0(x) \quad \text{in } (0, L_0), \end{cases}$$

such that the PDE in (2) is satisfied in Q_L^* in the distributional sense and the boundary conditions and (3) are fulfilled in $(0, T_*)$. Furthermore, either $T_* = T$, or at least one of the following identities hold:

$$\lim_{t \to T_*^-} L(t) = b \quad \text{or} \quad \lim_{t \to T_*^-} |L'(t)| = +\infty.$$

This is established in [1]; see also [2].

Free-boundary problems similar to (1)–(3) are motivated by many different applications:

- Solidification processes and, in particular, the so called *Stefan problem*, see [3,4].
- The analysis and computation of free surface flows, see [5–7].
- Fluid–solid interaction, see [8–10].
- Gas flow through porous media, see [11–13].
- Tumor growth and other problems from mathematical biology, see [14,15], etc.

Note that, for a fixed control v, (1) and (2) are not enough to identify the state (L, y). We need an additional information and this is furnished by (3). In many of these areas, this condition is completely natural. For instance, in tumor growth modeling, y can be viewed as a pressure (cells are pushed towards the low density regions) and (3) says that, on the moving tumor boundary, the growth speed is proportional to the pressure gradient. Of course, this is a version of the well known *Darcy's law* for porous media.

The main goal in this paper is to analyze the null controllability of (1)–(3). By definition, it will be said that (1)–(3) is *null-controllable* if, for any $y_0 \in H_0^1(0, L_0)$, there exist a control $v \in L^2(\omega \times (0, T))$, a function $L \in C^1([0, T])$ and an associated solution $y = y(x, t)$ satisfying (1)–(3) and

$$y(x, T) = 0, \quad x \in (0, L(T)). \tag{4}$$

Notice that, if L and (v, y) solve (1)–(4) and (for instance) $y(\cdot, T) \in H^2(0, L(T))$, then $L'(T) = 0$. Consequently, the null controllability of (1)–(3) is a useful property from the viewpoint of control theory: roughly speaking, if it is fulfilled, we only have to "work" during a finite time interval in order to get the desired behavior of the system.

The controllability of linear and nonlinear PDEs has been the objective of a lot of papers in the last years. In the context of the linear and semilinear heat equation, the main contributions have been obtained in [16–21]. On the other hand, to our knowledge, for parabolic free-boundary problems, controllability questions have not been considered in depth; see however [22,23].

In the sequel, C denotes a generic positive constant; C_0, C_1, etc. are other positive (specific) constants; when it makes sense, the extension by zero in space of any function f is denoted by f^*.

The main result in this paper is the following:

Theorem 1.1. *Let us assume that $L_0 > 0$, $T > 0$ and $0 < a < b < L_* < L_0 < B$ are given. Then (1)–(3) is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $y_0 \in H_0^1(0, L_0)$ and $\|y_0\|_{H_0^1(0,L_0)} \leq \varepsilon$, there exist L and (v, y), with*

$$L \in C^1([0, T]), \qquad L_* \leq L(t) \leq B,$$
$$v \in L^2(\omega \times (0, T)), \qquad y^* \in C^0([0, T]; H_0^1(0, B)),$$

that satisfy (1)–(3) and (4).

The plan of the proof is the following:

- First, we prove that there exists $\varepsilon > 0$ such that, whenever $\|y_0\|_{H_0^1(0,L_0)} \leq \varepsilon$, for each $\beta > 0$ there exist uniformly bounded L_β and (v_β, y_β) satisfying (1)–(3) and

  $$\|y_\beta(\cdot, T)\|_{L^2(0,L(T))} \leq \beta. \tag{5}$$

  This is achieved by solving an appropriate fixed point equation in a closed convex set $\mathcal{M} \subset C^1([0, T])$:

  $$L = \Lambda_\beta(L), \quad L \in \mathcal{M}.$$

- Then, we take limits as $\beta \to 0$. Thus, from the estimates deduced for L_β and (v_β, y_β), we see that, at least for a subsequence, we have

  $L_\beta \to L$ strongly in $C^1([0, T])$,

  $v_\beta \to v$ weakly in $L^2(\omega \times (0, T))$,

  $y_\beta^* \to y^*$ strongly in $C^0([0, T]; H_0^1(0, B))$,

  where (v, y) is a control–state pair satisfying (1)–(3) and (4).

The rest of this paper is organized as follows.

In Section 2, we prove that, for any $L \in C^1([0, T])$ satisfying

$$L_* \leq L \leq B, \quad L(0) = L_0, \tag{6}$$

the linear system (2) is approximately null-controllable. We also prove that the solutions to (2) satisfy an additional regularity property. Section 3 is devoted to prove Theorem 1.1. In Section 4, we present some additional comments and we give some indications on future work. Finally, in the Appendix we sketch the proof of a Carleman estimate.

2. Some controllability results for the classical heat equation in a non-cylindrical domain

In this section, we assume that $L_0 > 0$, $T > 0$ and $0 < a < b < L_* < L_0 < B$ are given.

We fix $y_0 \in H_0^1(0, L_0)$ and we assume that $L \in C^1([0, T])$ is a prescribed function satisfying (6); in particular, note that $L(t) > b$ for all $t \in [0, T]$.

Throughout this paper, we will use the notation

$$N_L := \|L'\|_\infty, \qquad N_0 := \|y_0\|_{H_0^1(0,L_0)}. \tag{7}$$

2.1. The problems and the results

Let us consider the linear system

$$\begin{cases} y_t - y_{xx} = v 1_\omega, \quad (x, t) \in Q_L, \\ y(0, t) = 0, \quad y(L(t), t) = 0, \quad t \in (0, T), \\ y(x, 0) = y_0(x), \quad x \in (0, L_0). \end{cases} \tag{8}$$

For every $v \in L^2(\omega \times (0, T))$ and every $y_0 \in H_0^1(0, L_0)$, there exists exactly one solution to (8), with

$$y, y_x, y_{xx}, y_t \in L^2(Q_L)$$

and, consequently,

$$y^* \in C^0([0, T]; H_0^1(0, B)).$$

Indeed, an appropriate change of variable allows to rewrite (8) as a similar problem for a parabolic PDE of the form

$$z_\tau - z_{\xi\xi} + h(\xi, \tau) z_\xi = w$$

in a cylindrical domain, with a bounded coefficient h and a square-integrable right hand side w; see the precise definitions of ξ and τ below, after (21).

The following controllability result is satisfied by (8):