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Convergence rate of nonlinear switched systems

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ABSTRACT

This paper is concerned with the convergence rate of the solutions of nonlinear switched systems.

We first consider a switched system which is asymptotically stable for a class of switching signals but not for all switching signals. We show that solutions corresponding to that class of switching signals converge arbitrarily slowly to the origin.

Then we consider analytic switched systems for which a common weak quadratic Lyapunov function exists. Under two different sets of assumptions we provide explicit exponential convergence rates for switching signals with a fixed dwell-time.

1. Introduction

Switched systems combine continuous dynamics with discrete events [1], and have attracted growing interest in recent years. In particular, problems related to the stability of switched systems have received a lot of attention [1,2]. Among these issues the convergence rate of solutions is of capital importance in practical situations. Indeed, knowing that the solutions converge to the origin is not enough if this information does not come with an estimate of the rate of decay of the solutions. It can be found in [3] some results concerning the growth rate of solutions in the case of discrete-time switched linear systems. These results make use of the joint spectral radius which characterizes the maximal growth rate of solutions. The paper [4] characterizes the marginal unstability of continuous-time linear switched systems. A linear switched system is said to be marginally unstable if the largest Lyapunov exponent is equal to zero and there exists an unbounded trajectory. The Largest Lyapunov exponent plays the same role as the joint spectral radius for discrete-time linear switched systems. In [4], the authors give a growth rate of the solutions which is polynomial in time. In the present paper, we are interested in the rate of decay of the solutions converging to zero.

A switched system is a parametrized family of vector fields together with a law selecting at each time which vector field is responsible for the evolution of the system. This law is in general a piecewise constant and right-continuous function of time and is called a switching signal or switching law. For these switching signals a solution of the switched system is merely a concatenation of integral curves of the different vector fields. It is a well-known fact that switching among asymptotically stable vector fields can lead to trajectories that escape to infinity [2]. On the other hand switching adequately between unstable vector fields may lead to trajectories that converge to the origin. In [2] Liberzon and Morse identify three basic problems related to the stability of switched systems. These problems are:

1. Find conditions guaranteeing that the switched system is asymptotically stable for all switching signals;
2. Identify the largest class of switching signals for which the switched system is asymptotically stable;
3. If the vector fields are not asymptotically stable, find at least one switching signal that makes the switched system asymptotically stable.

In the papers [5] and [6] (see also [7,8]) systems asymptotically stable for some class of switching signals but not for all switching signals are identified. For these systems two questions naturally arise. Does there exist a common convergence rate for all switching signals of the class under consideration? If the answer is negative is it possible to determine some convergence rate depending on a parameter, for instance a dwell-time?

The paper is an attempt to answer these questions. It is organized as follows: after having recalled some basic facts in Section 2 we show in Section 3 that if the asymptotic stability restricts to some class of switching signals, for instance dwell-time switching signals or nonchaotic ones [5,6], then the solutions corresponding to these switching signals converge arbitrarily slowly to the origin.

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Then we turn our attention to switched systems defined by a set of analytic vector fields that share a common weak Lyapunov function (this is the framework of the previous papers [5,6,8–10]). In Section 4 we consider homogeneous vector fields and quadratic weak Lyapunov functions and we prove exponential stability, with a computable convergence rate, for all switching signals with a fixed dwell-time. In Section 5 this result is extended to general nonlinear switched systems under the assumptions that the differentials of the vector fields at the origin are Hurwitz and that the Hessian of the Lyapunov function is positive definite. Some partial results about linear systems can be found in [11].

2. Basic definitions

2.1. The switched system

Let $\{f_1, \dots, f_p\}$ be a finite family of smooth vector fields of $\mathbb{R}^d$. Here and subsequently we assume the origin to be a common equilibrium of the vector fields.

Consider the switched system

$$\dot{x} = f_u(x), \quad x \in \mathbb{R}^d,\ u \in \{1, \dots, p\} \tag{S}$$

a switching signal, or switching law, is a piecewise constant (*i.e.* the number of switches is finite on any bounded interval) and right-continuous function $t \longmapsto u(t)$ from $[0, +\infty)$ to $\{1, \dots, p\}$. Let us denote by $\mathfrak{S}$ the set of all switching signals.

Fixing such a switching signal $u \in \mathfrak{S}$ gives rise to a time-varying differential equation

$$\dot{x} = f_{u(t)}(x), \quad x \in \mathbb{R}^d.$$

Let us denote by $\Phi_u^t(x)$, $t \geq 0$, the solution of the switched system for the switching signal u and for the initial condition $x \in \mathbb{R}^d$. Let also $\Phi_i^t(x)$ stand for the flow of f_i which corresponds to the case where the switching signal is constant and equal to i. The switching signals being piecewise constant a solution of the switched system is nothing but a concatenation of integral curves of the vector fields f_i, $i = 1, \dots, p$. In the literature, one may encounter switching signals that are assumed to be merely measurable. In this context, a solution is an absolutely continuous curve that satisfies the differential equation almost everywhere.

2.2. Stability of switched systems

Here we introduce the concept of stability we are interested in and the notion of convergence rate.

Definition 2.1. Let $\mathcal{U}$ be a family of switching signals. The switched system (S) is said to be globally asymptotically stable over the class $\mathcal{U}$ if:

1. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall u \in \mathcal{U}, \|x\| \leq \delta \Longrightarrow \forall t \geq 0,\ \left\|\Phi_u^t(x)\right\| \leq \varepsilon$.
2. For any $u \in \mathcal{U}$ and any $x \in \mathbb{R}^d$, the solution $\Phi_u^t(x)$ converges to zero.

Definition 2.2. The switched system (S) is said to be GUAS (Globally Uniformly Asymptotically Stable) if

$$\forall \varepsilon, \delta > 0, \exists T > 0 \quad \text{s.t.} \quad \forall u \in \mathfrak{S}, \|x\| \leq \delta \Longrightarrow \forall t \geq T,$$
$$\|\Phi_u^t(x)\| \leq \varepsilon.$$

Here the word "*uniform*" means that there exists a common rate of decay to all the switching signals. One can give an equivalent characterization of the GUAS property which involves the rate of decay of the solutions. The proof of this equivalence can be found in [12] in the context of systems with disturbances.

Definition 2.3. A continuous function $\alpha : [0, +\infty) \longrightarrow \mathbb{R}_+$ is said to be of class $\mathcal{K}$ (resp. $\mathcal{K}_\infty$) if $\alpha(0) = 0$ and it is increasing (resp. $\alpha(0) = 0$ and it is increasing to $+\infty$).

A continuous function $\beta : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}_\infty$ and for each $r \in \mathbb{R}_+$ the function $\beta(r, \cdot)$ decreases to 0.

Proposition 2.1. *The switched system is GUAS if and only if there exists a class $\mathcal{KL}$ function β such that for all switching signals u and all initial conditions $x \in \mathbb{R}^d$,*

$$\forall t \geq 0, \quad \|\Phi_u^t(x)\| \leq \beta(\|x\|, t).$$

The function β provides a convergence rate of the solutions to the equilibrium.

In particular, if there exist two positive constants C and μ such that $\beta(r, t) = Cre^{-\mu t}$ then the convergence rate is said to be exponential.

It is well-known that as soon as an 1-homogeneous system (that is $f_i(\lambda x) = \lambda f_i(x)$) is GUAS the convergence rate is exponential [13]. Moreover, the GUAS property of 1-homogeneous switched systems is equivalent to the local attractivity of the origin *i.e.* all solutions near the origin converge to zero.

2.3. Classes of switching signals

In this section, the main classes of switching signals that can be found in the literature [5,13] are introduced.

Definition 2.4. A switching signal is said to have a *dwell-time* if the duration between two discontinuities is bounded below by some positive constant called the dwell-time. We denote by $\mathfrak{S}_{\mathrm{d}}$ the set of all switching signals which admit a dwell-time. For any positive δ, we also denote by $\mathfrak{S}_{\mathrm{d}}[\delta]$ the set of all switching signals for which the dwell-time is at least δ.

A switching signal that presents a finite number of discontinuities has a dwell-time.

Definition 2.5. A switching signal is said to have an *average dwell-time* if there exist two positive numbers δ and N_0 such that for any $t \geq 0$ the number $N(t, t+\tau)$ of discontinuities of u on the interval $]t, t+\tau[$ satisfies

$$N(t, t+\tau) \leq N_0 + \frac{\tau}{\delta}.$$

The number δ is called the *average dwell-time* and N_0 the *chatter bound*. The set of all switching signals that have an average dwell-time will be denoted by $\mathfrak{S}_{\mathrm{ad}}$.

Definition 2.6. A switching signal is said to have a *persistent dwell-time* if there exists a sequence of times $(t_k)_{k \geq 0}$ increasing to $+\infty$ and two positive numbers δ and T such that u is constant on $[t_k, t_k + \delta)$ and $t_{k+1} - t_k \leq T$ for all $k \geq 0$. The constant δ is called the *persistent dwell-time* and T the *period of persistence*. The set of all switching signals that have a persistent dwell-time will be denoted by $\mathfrak{S}_{\mathrm{perst}}$.

Notice that a constant switching signal has a persistent dwell-time and a period of persistence equal to any positive number.

Definition 2.7 (*See [5]*). A switching signal is said to be chaotic if there exists a positive constant τ and a sequence $([t_k, t_k + \tau])_{k \geq 0}$ of intervals that satisfies the following conditions:

1. $t_k \longrightarrow_{k \to +\infty} +\infty$;
2. For all $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the switching signal is constant on no subinterval of $[t_k, t_k + \tau]$ of length greater than or equal to ε.