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A note on observability of Boolean control networks[☆]

Daizhan Cheng[a,b,*], Hongsheng Qi[a], Ting Liu[a], Yuanhua Wang[b]

[a] Key Laboratory of Systems & Control, Academy of Mathematics & Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

[b] School of Control Science and Engineering, Shandong University, Ji'nan 250061, PR China

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ABSTRACT

The observability of Boolean control networks is investigated. The pairs of states are classified into three classes: (i) diagonal, (ii) h-distinguishable, and (iii) h-indistinguishable. For h-indistinguishable pairs, we construct a matrix $\mathcal{W}$ called the transferable matrix, which indicates the control-transferability among h-indistinguishable pairs. Modifying $\mathcal{W}$ yields a Boolean matrix $\mathcal{U}^0$, which is used as the initial matrix for an iterative algorithm. After finite iterations a stable $\mathcal{U}^*$ is reached, which is called the observability matrix. It is proved that a Boolean control network is observable, if and only if, the last column of $\mathcal{U}^*$, $\mathrm{Col}_{r+1}(\mathcal{U}^*) = \mathbf{1}_r$. Some numerical examples are presented.

1. Introduction

The Boolean networks (BNs) were first proposed by S. Kauffman to describe genetic regulatory networks [1]. Since then the BN has attracted a considerable attention from systems biology, physics as well as systems science. In 2001, [2] pointed out that the genetic regulatory networks have input(s) and output(s), and they can be described as Boolean control networks (BCNs). Then the investigation of BCNs increases [3–6]. But during this period, most of the research were concentrated on control only. As pointed out by [5] that "One of the major goals of systems biology is to develop a control theory for complex biological systems". But because the genetic networks are logical and there were shortage of proper tools to deal with logical dynamic systems, the results on Boolean control networks (BCNs) were limited.

Using semi-tensor product (STP) of matrices, an algebraic state space approach to BNs and BCNs was proposed [7,8]. It stimulates the research on BNs and BCNs. We refer the reader to [9] for several dynamic and/or control problems of BNs, and to [10] for STP.

The controllability and observability are two fundamental problems in the control of BNs as well as in the control theory. The controllability of various types of BNs has been solved neatly. For instance, [7,11] solved the controllability of standard form of BCNs; the controllability of state restricted BCNs was solved in [12]; the controllability of probabilistic BCNs was solved in [13,14]. The controllability of time-varying BCNs [15], higher order BCNs [16,17], switched BCNs [18,19], time-delay BCNs [20,21], and periodic BCNs [22], etc., has also been investigated.

Similarly, the observability of BCNs has also been widely investigated. Though there is no dual relationship between controllability and observability such as for linear systems, as a convention, sometimes the observability of BCNs is still discussed simultaneously with controllability [7,21,23–26].

Unlike the controllability, the observability of BCNs has various definitions, and for the most general (sharp) definition, the necessary and sufficient condition was still not known until [27], see also [28].

First of all, [27] discussed four different definitions of observability in the recent literature. We first cite these four definitions in a uniform way, which might be different from the original ones in statement, but have been proved in [27] that the following four definitions are equivalent to their original ones.

Definition 1.1. A BCN is observable, if

(D1) [7] for any initial state x_0 there exists an input sequence $\{u_0, u_1, \ldots\}$ such that for any $\bar{x}_0 \neq x_0$ the corresponding output sequences $(y_0, y_1, \ldots) \neq (\bar{y}_0, \bar{y}_1, \ldots)$;

(D2) [11] for any two distinct states x_0, $\bar{x}_0$ there is an input sequence $\{u_0, u_1, \ldots, u_p\}$, $p \in \mathbb{Z}_+$, such that the corresponding output sequences $(y_0, y_1, \ldots, y_p) \neq (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_p)$;

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* Corresponding author at: Key Laboratory of Systems & Control, Academy of Mathematics & Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China.

E-mail address: dcheng@iss.ac.cn (D. Cheng).

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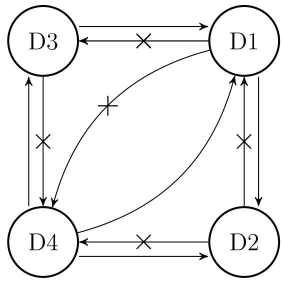

Fig. 1. The relationships of D1–D4.

(D3) [23] there exists an input sequence $\{u_0, u_1, \dots, u_p\}$, $p \in \mathbb{Z}_+$, such that for any two distinct x_0, $\bar{x}_0$, the corresponding output sequences $(y_0, y_1, \dots, y_p) \neq (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_p)$;

(D4) [24] for any two distinct states x_0, $\bar{x}_0$ and for any input sequence $\{u_0, u_1, \dots\}$, the corresponding output sequences $(y_0, y_1, \dots) \neq (\bar{y}_0, \bar{y}_1, \dots)$.

The relationship among these four definitions is described in Fig. 1 [27].

In Fig. 1 "$\rightarrow$" means implication and "$\not\rightarrow$" means not implication. Note that the "implication" means that if a BCN satisfies the preceding definition it also satisfies the following one. From Fig. 1 it is clear that D2 is the most sensitive (sharp) one. So as proposed by [27], we may take D2 as the standard one and concentrate on this definition. Hereafter, the observability of BCNs we concerned will be the one specified by D2.

In [11] only a sufficient condition was provided, while other papers deal with various other kinds of observability. Hence, the necessary and sufficient condition for observability of Boolean networks was still unknown until [27].

By resorting to formal language and finite automata, [27] (refer also to [28]) presents a necessary and sufficient condition for the observability of BCNs. Their result is like this: For each pair of distinct states $(x_0, \bar{x}_0)$, an algorithm is provided to construct a deterministic finite automata (DFA), denoted by $A_{(x_0,\bar{x}_0)}$. Then a system is not observable, if and only if, there is a pair of distinct states $(x_0, \bar{x}_0)$, such that the corresponding DFA, $A_{(x_0,\bar{x}_0)}$ can recognize its corresponding alphabet.

The result provided by [27] is the first theoretically verifiable necessary and sufficient condition for the observability of BCNs. But its computational complexity is a severe problem. As described in the paper, it is necessary to draw a DFA for each pair of h-indistinguishable pair of states, and then verify its recognizable languages. It can be practically done only for very small toy systems. Moreover, the knowledge about formal language and finite automata is required to understand their technique.

The purpose of this paper is to give an alternative set of necessary and sufficient conditions for the observability of BCNs. The necessary and sufficient conditions are easily verifiable and do not involve any additional auxiliary machines such as finite automata or so. Using the transition matrix of h-indistinguishable matrix $\mathcal{W}$ we construct a Boolean matrix, $\mathcal{U}^0$. That is a matrix with entries in $\{0, 1\}$. Then an algorithm is proposed to perform an iteration on $\{\mathcal{U}^i | i = 0, 1, \dots\}$. After finite iterations a fixed matrix $\mathcal{U}^*$, called the observability matrix, will be reached. It is proved that the BCN is observable, if and only if, the last column of $\mathcal{U}^*$, which is the set of distinguishable indices of each rows respectively, is $\mathrm{Col}_{r+1}(\mathcal{U}^*) = \mathbf{1}_r$.

Though the approach seems completely different from [27], the initial idea was motivated by [27].

The paper is organized as follows: Section 2 presents some preliminaries. It consists of two subsections: one is a brief introduction to the semi-tensor product of matrices, and the other is for the algebraic state space representation of logical dynamic systems. Section 3 studies the observability of BCN. The h-indistinguishable matrix $\mathcal{W}$ is constructed. Using it, the algorithm is introduced. Then the main result is obtained as a necessary and sufficient condition. In Section 4 some illustrative examples are presented to demonstrate the algorithm and the main result. Some related topics are discussed in Section 5 as the concluding remarks.

2. Preliminaries

2.1. Semi-tensor product of matrices

This subsection gives a brief review for STP. The readers can refer to [10] for details.

First, we give some notations:

- $\mathbb{Z}_+$: the set of non-negative numbers.
- $\mathbf{1}_n = [\underbrace{1, \dots, 1}_{n}]^T$.
- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the ith column (row) of M.
- $\mathcal{D} := \{0, 1\}$.
- δ_n^i: the ith column of the identity matrix I_n.
- $\Delta_n := \{\delta_n^i | i = 1, \dots, n\}$, $\Delta := \Delta_2$.
- A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L, denoted by $\mathrm{Col}(L)$, are of the form δ_m^k, $1 \le k \le m$. That is,

  $\mathrm{Col}(L) \subset \Delta_m$.

  Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.
- If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_r}]$. For the sake of brevity, it is briefly denoted as $L = \delta_n[i_1, i_2, \dots, i_r]$.

Definition 2.1. Let $M \in \mathcal{M}_{m\times n}$ and $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product (STP) of M and N, denoted by $M \ltimes N$, is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \tag{1}$$

where $\otimes$ is the Kronecker product.

When $n = p$, the STP coincides with the conventional matrix product. So the STP is a generalization of conventional matrix product. Fortunately, it keeps all the properties of the conventional matrix product unchanged. We, therefore, omit the symbol "$\ltimes$" mostly. In addition, it has some new properties. The following property is frequently used in the sequel.

Proposition 2.2. *Let $X \in \mathbb{R}^m$ be a column and M be any matrix. Then*

$$X \ltimes M = (I_m \otimes M)X. \tag{2}$$

Definition 2.3 ([29]). $M \in \mathcal{M}_{m\times p}$, $N \in \mathcal{M}_{n\times p}$. The Khatri–Rao product of M and N is defined as

$$\begin{aligned} M * N := [&\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N), \dots, \\ &\mathrm{Col}_p(M) \ltimes \mathrm{Col}_p(N)] \in \mathcal{M}_{mn\times p}. \end{aligned} \tag{3}$$

2.2. Algebraic state space representation of Boolean networks

Definition 2.4. 1. A function $f : \mathcal{D}^n \to \mathcal{D}$ is called a Boolean function. It can be expressed as

$$y = f(x_1, x_2, \dots, x_n), \quad y, x_1, \dots, x_n \in \mathcal{D}. \tag{4}$$