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A study on the numerical convergence of the discrete logistic map

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ABSTRACT

Clocking convergence is an important tool for investigating various aspects of iterative maps, especially chaotic maps. In this work, we revisit to the numerical convergence of the discrete logistic map $x_{n+1} = rx_n(1 - x_n)$ gauged with a finite computational accuracy. Most of the previous studies of the discrete logistic map have been made for $r \in [3, 4]$ and $r \in [-2, -1]$ due to the rich complexity of the map in these regions. In this work, we consider regions with simple fixed points, i.e. $r = [-1, 3]$ for which no particular geometric structures are known, as well as the period-doubling regions. We numerically investigate the speed of convergence in these regions to expose underlying complexity. The convergence speed is mapped to the phase space with different finite precisions. Patterns generated through this map are investigated over r. Numerical results show that there exists an interesting geometric pattern in $r \in [-1, 3]$ when convergence is gauged with a finite computational precision and also show that this pattern cascades into the period-doubling areas.

1. Introduction

Clocking convergence has been an important tool for investigating various aspects of iterative maps such as chaos, attractors, etc. [17]. For example, detecting chaos is highly sensitive to the convergence rate of the map with a given finite computation accuracy. An accurate calculation of the correlation dimension for the detection of chaos is, in general, difficult because of the slow convergence rate and demands a huge computing capacity [6,17]. The onset of chaos shows different behaviors depending on the different convergence types [21]. Measuring the Lyapunov exponent also relies on the convergent trajectory within the finite trajectory. In [19], a numerical study has been made on convergence of Lyapunov exponent estimates for the logistic map and the anomalous precision of the Lyapunov exponent estimates has been discussed. In [13], the rate of convergence to the critical attractor of dissipative maps has been considered. Such convergence behavior is also important in synchronizing chaotic maps [15] and its applications to such as communications, cryptography, etc. [8,10,11]. Among various discrete maps, the discrete logistic maps have been studied intensively and many interesting features are now well known and understood [3,4,14,18]. Recently the logistic maps have been also investigated in various context such as in cryptography [9–11], chaotic measures and complexity [1,20], a coupled system [7], delayed equations [16], and applications in control [5].

Most of these convergence studies have been made for the regions of the phase space where the complexity has its rich structures such as bifurcation, chaos, etc. The main purpose of this paper is to revisit to the discrete logistic map and investigate

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the convergent behavior numerically in the plain convergent region with a finite computational precision. Compared to the complex behaviors in the chaotic region of the logistic map, the plain convergent region has not been well studied as it has a deterministic and simple dynamic behavior in the region. In this paper, we will show that there can be found interesting dynamic structures in this area as well by examining the numerical convergence in this region. The numerical convergence is mapped in the phase space and the pattern generated by this map is investigated.

In this paper, we consider the following discrete map $f : \mathbb{X} \mapsto \mathbb{R}, x_n \in \mathbb{X} \subset \mathbb{R}\ \forall n = 0, 1, \ldots,$

$$x_{n+1} = f(x_n). \tag{1}$$

For later convenience, we also define the inverse map f^{-1} which is not necessarily unique, but $f^{-1}(x_n) \in \mathbb{X}\ \forall n \in \mathbb{Z}^*$. Convergence of the map Eq. (1) is clocked with a finite computational precision. Assume that the given sequence is *numerically convergent*, that is, there exists a positive index $k < \infty, k \in \mathbb{Z}^+$ such that $|x_l - x_{l-1}| \leqslant \epsilon,\ \forall l \geqslant k$, where ϵ is a given strictly positive real number. The parameter ϵ is a finite computational precision. For the current study, we choose ϵ as low as machine precision. In our study, machine precision is given by $\epsilon \sim 10^{-16}$ and this value is the lowest limit used in this work. The convergent index k is the number of discrete time steps it takes until the numerical convergence is achieved. The convergent behavior is mapped onto the phase space of the initial data x_0 and the parameter r.

The logistic map is a simple non-linear population dynamics model with very rich complexity in quadratic form

$$x_{n+1} := f(x_n, r) = r x_n (1 - x_n), \tag{2}$$

where x_{n+1} denotes the normalized population at the discrete time step $n + 1$ with the initial population x_0 [2,12,18].

For the logistic map f, the inverse map f^{-1} is not necessarily unique. Since Eq. (2) was originally introduced for the population model, the normalized population x_n is non-negative and

$$x_n \in \mathbb{X} = [0, 1] \quad \forall n \in \mathbb{Z}^*$$

and consequently we have

$$0 \leqslant r \leqslant 4.$$

The negative value of r, however, can also provide a mathematically meaningful map as shown in Fig. 1 although no negative population is defined in the real world. For $0 \leqslant r \leqslant 1, x_n$ converges to 0 for any initial value $x_0 \in \mathbb{X}$. For $1 < r \leqslant 3,\ x_n$ converges to $x^*(r) = (r-1)/r$. As r increases past 3, x_n starts skipping between two values, then four, and so on towards the complete chaos. This is most clearly displayed in the bifurcation diagram in Fig. 1. Fig. 1 also shows that a similar bifurcation diagram is obtained in the region of $-2 \leqslant r \leqslant 0$.

The fixed points x^* for a given r are given by

$$x^* = 0 \quad \text{or} \quad \frac{r-1}{r}.$$

For $1 < r < 3$, there are two fixed points $x^*(r) = 0$ becomes unstable while $x^*(r) = (r-1)/r$ is stable. For $r \in [1, 2]$ convergence is fast and the initial data x_0 quickly reaches the fixed point. If r is between 2 and 3, convergence is slower than that in $r \in [1, 2]$ but any initial data x_0 reaches the fixed point. As the bifurcation diagram implies, the interesting features of the

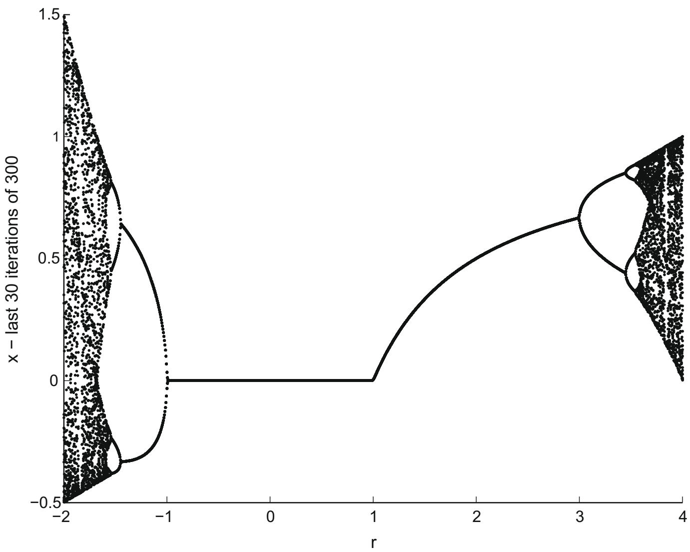

Fig. 1. Bifurcation diagram of the logistic map. The figure shows the last 30 maps from the total 300 iterations for a given r, $-2 \leqslant r \leqslant 4$.