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# Unsteady Couette flows in a second grade fluid with variable material properties

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**Abstract**

Fluid solid mixtures are generally considered as second grade fluids and are modeled as fluids with variable physical parameters. Thus, an analysis is performed for a second grade fluid with space dependent viscosity, elasticity and density. Two types of time-dependent flows are investigated. An eigen function expansion method is used to find the velocity distribution. The obtained solutions satisfy the boundary and initial conditions and the governing equation. Remarkably some exact analytic solutions are possible for flows involving second grade fluid with variable material properties in terms of trigonometric and Chebyshev functions.


*Keywords:* Second grade fluid; Variable material constants; Analytic solutions; Chebyshev polynomials

## 1. Introduction

Fluid solid mixtures are generally considered to behave like non-Newtonian fluids. This type of fluids occur in pneumatic and hydraulic transport of solids and thus have many industrial applications. A specific research area in this direction is the use of coal based slurries which requires the analysis of various transport processes in non-Newtonian fluids. If we consider the mixture as a single homogeneous continuum then it is reasonable to model the fluid with variable physical properties.

Viscoelastic flows arise in disparate processes in engineering, science, and biology for example, in polymer processing, coating, ink-jet printing, microfluidics, geological flows in the earth mantle, hemodynamics, flow of synovial fluid in joints, and many others. Modeling viscoelastic flows is important for understanding and predicting the behavior of processes, for designing optimal flow configurations and for selecting operating conditions.

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Due to complexity of viscoelastic fluids, there is no single model which describes all of their properties. Many models for such fluids have been proposed. Amongst the many models, there is a second grade model which is the most popular. This is particularly so due to the fact that one can reasonably hope to obtain the analytic solution. Mention many be made to the interesting studies of second grade fluids in Refs. [1–12].

In all the studies mentioned above, the density, viscosity and elasticity of the fluid have been considered constant. Such considerations are not adequate in many engineering applications. Thus, the main objective of the present work is to study the unsteady flows of a second grade fluid when density, dynamic viscosity and elasticity are not constant, i.e. these depend on the space variable. To the knowledge of the authors such study in a second grade fluid does not seem to have been undertaken. Two problems of unidirectional flow involving second grade fluid are considered. Interestingly, exact analytical solution in each case is obtained using the eigen function expansion method.

There are only a few cases which allow the analytic solution of non-Newtonian fluids. The variable viscosities have wide range of applications and one particular example is of mixture fluid, responsible for a changing viscosity. It is hoped that a new dimension of variable parameters will help to understand the fluids with solid fluid mixture.

## 2. Governing equations

The constitutive assumption for the fluid of second grade is of the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

in which $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ is the spherical stress and $\mu$, $\alpha_1$ and $\alpha_2$ are material constants. They denote, respectively, the dynamic viscosity, elasticity and cross-viscosity. $\mathbf{A}_i$ $(i = 1, 2)$ are the first two Rivlin–Ericksen tensors defined by

$$\mathbf{A}_1 = \mathbf{L} + \overset{*}{\mathbf{L}}, \quad \mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1\mathbf{L} + \overset{*}{\mathbf{L}}\,\mathbf{A}_1, \quad \mathbf{L} = \nabla\mathbf{V}, \tag{2}$$

where $\mathrm{d}/\mathrm{d}t$ is the material derivative and $(^*)$ signifies the matrix transpose. Further, the critical review of Dunn and Rajagopal [13] already gives a concise discussion about the requirement of model (1) to be compatible with thermodynamics. For unidirection flow, we take the velocity fluid

$$\mathbf{V} = (u(y,t), 0, 0), \tag{3}$$

where $u$ is the velocity in the $x$ direction. Moreover, what we require is that, the viscosity, elasticity and density are not constants but varies with respect to the space variable $y$. Using Eq. (3), the continuity equation is satisfied identically and the momentum equation in the absence of pressure gradient yields

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial y}\left(\alpha_1\frac{\partial u}{\partial y}\right)\right] - \rho(y)\frac{\partial u}{\partial t} = 0. \tag{4}$$

In what follows is that we present the two independent problems taking different space dependences of the physical parameters.

## 3. Couette flows

**Case 1** (*When* $\mu$, $\alpha_1$, and $\rho$ are linear functions of $y$). Let us consider the second grade fluid with variable viscosity, elasticity and density between two rigid plates which are at rest initially. The plate at $y = h\mathrm{e}^{-\pi}$ is fixed and the fluid motion starts suddenly due to a constant velocity $U_0$ of the plate at $y = h\mathrm{e}^{-\pi}$. The governing differential equation is (4) with the following boundary and initial conditions.

$$\begin{aligned} &u(h\mathrm{e}^{-\pi}, t) = 0, \qquad \text{for all } t,\\ &u(h\mathrm{e}^{\pi}, t) = U_0, \qquad \text{for } t > 0, \\ &u(y, 0) = 0, \qquad \text{for } h\mathrm{e}^{-\pi} < y < h\mathrm{e}^{\pi}. \end{aligned} \tag{5}$$