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Control of 2 × 2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking[☆]

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ABSTRACT

We consider the problems of trajectory generation and tracking for general 2 × 2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the trajectory generation problem via backstepping. The reference input, which generates the desired output, incorporates integral operators acting on advanced and delayed versions of the reference output with kernels which were derived by Vazquez, Krstic, and Coron for the backstepping stabilization of 2 × 2 linear hyperbolic systems. We apply our approach to a wave PDE with indefinite in-domain and boundary damping. For tracking the desired trajectory we employ a PI control law on the tracking error of the output. We prove exponential stability of the closed-loop system, under the proposed PI control law, when the parameters of the plant and the controller satisfy certain conditions, by constructing a novel "non-diagonal" Lyapunov functional. We demonstrate that the proposed PI control law compensates in the output the effect of in-domain and boundary disturbances. We illustrate our results with numerical examples.

1. Introduction

Control of 2 × 2 systems of first-order hyperbolic PDEs is an active area of research since numerous processes can be modeled with this class of PDE systems. Among various applications, 2 × 2 systems model the dynamics of traffic [1,2], hydraulic [3–6], as well as gas pipeline networks [7], and the dynamics of transmission lines [8].

Several articles are dedicated to the control and analysis of 2 × 2 linear [3,9,5,10] [11–13] and nonlinear [14–19] systems. Results for the control of $n \times n$ systems also exist [20–23]. Algorithms for disturbance rejection in 2 × 2 systems are recently developed [24, 25]. The motion planning problem is solved in [26,27], for a class of 2 × 2 systems and in [28,29] for a class of wave PDEs. Perhaps the most relevant results to the present article are the results in [5], dealing with the Lyapunov-based output-feedback control of 2 × 2 linear systems, the results in [12], dealing with the backstepping stabilization of 2 × 2 linear systems, and the results in [27], dealing with the motion planning for a class of 2 × 2 systems.

In this paper, we are concerned with the trajectory generation and tracking problems for general 2 × 2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the motion planning problem for this class of systems employing backstepping (Section 2.1). Specifically, we start from a simple transformed system, namely, a cascade of two first-order hyperbolic PDEs, for which the motion planning problem can be trivially solved. We then apply an inverse backstepping transformation to derive the reference trajectory and reference input for the original system. Our approach is different than the one in [12], in that we use backstepping for trajectory generation rather than stabilization, and the one in [27], in that we employ a different conceptual idea to a different class of systems. The idea of the backstepping-based trajectory generation for PDEs, which was conceived in [30], is applied to a beam PDE in [31] and the Navier–Stokes equations in [32], and is recently extended to general $n \times n$ linear hyperbolic systems in [22]. We apply this methodology to a wave PDE with indefinite in-domain and boundary damping by transforming (see, for example, [33]) the wave PDE to a 2 × 2 linear hyperbolic system coupled with a first-order ODE (Section 2.2).

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We then employ a PI control law for the stabilization of the error system, namely, the system whose state is defined as the difference between the state of the plant and the reference trajectory. We prove exponential stability in the L_2 norm of the closed-loop system by constructing a Lyapunov functional which incorporates cross-terms between the PDE states of the system and the ODE state of the controller, when the parameters of the system and the controller satisfy certain conditions (Section 3.1). Our result differs than the result in [5] in that we employ PI control on an output of the system in the Riemann coordinates and we construct a non-diagonal Lyapunov functional for proving closed-loop stability. We demonstrate that the proposed PI control law is capable of compensating in the output the effect of additive disturbances affecting the boundary or the interior of the PDE domain (Section 3.2). We present several examples, for the illustration of our methodologies, including a simulation example dealing with the generation of a sinusoidal reference trajectory for a wave PDE (Section 4.1) and a simulation example of a system tracking a sinusoidal reference output (Section 4.2).

2. Trajectory generation using backstepping

2.1. General 2 × 2 linear hyperbolic systems

We consider the following system

$$z_t^1 + \varepsilon_1(x) z_x^1 = c_1(x) z^1 + c_2(x) z^2 \tag{1}$$

$$z_t^2 - \varepsilon_2(x) z_x^2 = c_3(x) z^1 + c_4(x) z^2, \tag{2}$$

under the boundary conditions

$$z^1(0,t) = q z^2(0,t) \tag{3}$$

$$z^2(1,t) = S(t) \tag{4}$$

$$z^2(0,t) = y(t), \tag{5}$$

where $t \in [0, +\infty)$ is the time variable, $x \in [0, 1]$ is the spatial variable, y is the output of the system, and S is the control input. The functions $\varepsilon_1, \varepsilon_2$ belong to $C^2([0,1])$ and satisfy $\varepsilon_1(x), \varepsilon_2(x) > 0$, for all $x \in [0,1]$, and the functions c_i, $i = 1,2,3,4$ belong to $C^1([0,1])$.

Defining the change of variables (see, for example, [3])

$$\chi_1(x) = \exp\left(-\int_0^x \frac{c_1(s)}{\varepsilon_1(s)} ds\right) \tag{6}$$

$$\chi_2(x) = \exp\left(\int_0^x \frac{c_4(s)}{\varepsilon_2(s)} ds\right) \tag{7}$$

$$\chi(x) = \frac{\chi_1(x)}{\chi_2(x)}, \tag{8}$$

and the new coordinates

$$u = \chi_1(x) z^1 \tag{9}$$

$$v = \chi_2(x) z^2, \tag{10}$$

system (1)–(5) is transformed into the following system

$$u_t + \varepsilon_1(x) u_x = \gamma_1(x) v \tag{11}$$

$$v_t - \varepsilon_2(x) v_x = \gamma_2(x) u, \tag{12}$$

with

$$\gamma_1(x) = \chi(x) c_2(x) \tag{13}$$

$$\gamma_2(x) = \chi^{-1}(x) c_3(x). \tag{14}$$

The boundary conditions become

$$u(0,t) = q v(0,t) \tag{15}$$

$$v(1,t) = U(t) \tag{16}$$

$$v(0,t) = y(t), \tag{17}$$

where the original control variable satisfies

$$U = \chi_2(1) S. \tag{18}$$

We aim at designing a reference control input $U^r(t)$ such that the output $y(t)$ follows a given reference trajectory $y^r(t)$, for $t \geq 0$. For achieving this we need first to construct the reference trajectory $(u^r(x,t), v^r(x,t))$ that satisfies (11), (12), (15), and (17) with $y(t) = y^r(t)$. The trajectory generation problem is solvable when the initial data (u^0, v^0) match the reference trajectory, i.e., when $u^0(x) = u^r(x,0)$ and $v^0(x) = v^r(x,0)$ (and hence, the initial data belong to the same space with $u^r(x,0)$ and $v^r(x,0)$).

Theorem 1. *Let $y^r \in C^1(\mathbb{R})$ be uniformly bounded. The functions*

$$\begin{aligned} u^r(x,t) &= q y^r(t - \Phi_1(x)) + \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)} y^r(t - \Phi_1(x) + \Phi_1(\xi)) d\xi \\ &+ q \int_0^x L^{\alpha\alpha}(x,\xi) y^r(t - \Phi_1(\xi)) d\xi \\ &+ \int_0^x L^{\alpha\alpha}(x,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(\xi) + \Phi_1(\zeta)) d\zeta d\xi \\ &+ \int_0^x L^{\alpha\beta}(x,\xi) y^r(t + \Phi_2(\xi)) d\xi \end{aligned} \tag{19}$$

$$\begin{aligned} v^r(x,t) &= y^r(t + \Phi_2(x)) + q \int_0^x L^{\beta\alpha}(x,\xi) y^r(t - \Phi_1(\xi)) d\xi \\ &+ \int_0^x L^{\beta\alpha}(x,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(\xi) + \Phi_1(\zeta)) d\zeta d\xi \\ &+ \int_0^x L^{\beta\beta}(x,\xi) y^r(t + \Phi_2(\xi)) d\xi \end{aligned} \tag{20}$$

$$\begin{aligned} U^r(t) &= y^r(t + \Phi_2(1)) + q \int_0^1 L^{\beta\alpha}(1,\xi) y^r(t - \Phi_1(\xi)) d\xi \\ &+ \int_0^1 L^{\beta\alpha}(1,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(\xi) + \Phi_1(\zeta)) d\zeta d\xi \\ &+ \int_0^1 L^{\beta\beta}(1,\xi) y^r(t + \Phi_2(\xi)) d\xi, \end{aligned} \tag{21}$$

where

$$\Phi_1(x) = \int_0^x \frac{1}{\varepsilon_1(s)} ds \tag{22}$$

$$\Phi_2(x) = \int_0^x \frac{1}{\varepsilon_2(s)} ds \tag{23}$$

$$f(x) = \begin{cases} \varepsilon_2(0) K^{uv}(x,0), & \text{if } q = 0 \\ 0, & \text{if } q \neq 0, \end{cases} \tag{24}$$

and $L^{\alpha\alpha}, L^{\alpha\beta}, L^{\beta\alpha}, L^{\beta\beta}, K^{uv}$ are the solutions of the following equations

$$\varepsilon_2(x) L_x^{\beta\alpha} - \varepsilon_1(\xi) L_\xi^{\beta\alpha} = \varepsilon_1'(\xi) L^{\beta\alpha} - \gamma_2(x) L^{\alpha\alpha} \tag{25}$$

$$\varepsilon_2(x) L_x^{\beta\beta} + \varepsilon_2(\xi) L_\xi^{\beta\beta} = -\varepsilon_2'(\xi) L^{\beta\beta} - \gamma_2(x) L^{\alpha\beta} \tag{26}$$

$$\varepsilon_1(x) L_x^{\alpha\alpha} + \varepsilon_1(\xi) L_\xi^{\alpha\alpha} = -\varepsilon_1'(\xi) L^{\alpha\alpha} + \gamma_1(x) L^{\beta\alpha} \tag{27}$$

$$\varepsilon_1(x) L_x^{\alpha\beta} - \varepsilon_2(\xi) L_\xi^{\alpha\beta} = \varepsilon_2'(\xi) L^{\alpha\beta} + \gamma_1(x) L^{\beta\beta} \tag{28}$$

$$\varepsilon_1(x) K_x^{uu} + \varepsilon_1(\xi) K_\xi^{uu} = -\varepsilon_1'(\xi) K^{uu} - \gamma_2(x) K^{uv} \tag{29}$$

$$\varepsilon_1(x) K_x^{uv} - \varepsilon_2(\xi) K_\xi^{uv} = \varepsilon_2'(\xi) K^{uv} - \gamma_1(x) K^{uu}, \tag{30}$$

with the boundary conditions

$$L^{\beta\alpha}(x,x) = -\frac{\gamma_2(x)}{\varepsilon_1(x) + \varepsilon_2(x)} \tag{31}$$