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Computing the regularization of a linear differential–algebraic system

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ABSTRACT

We study the regularization problem for linear differential–algebraic systems. As an improvement of former results we show that any system can be regularized by a combination of state-space and input-space transformations, behavioral equivalence transformations and a reorganization of variables. The additional state feedback which is needed in earlier publications is shown to be superfluous. We provide an algorithmic procedure for the construction of the regularization and discuss computational aspects.

1. Introduction

We study linear descriptor systems given by differential–algebraic equations (DAEs) of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t) \tag{1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$. The set of systems (1) is denoted by $\Sigma_{l,n,m}$ and we write $[E, A, B] \in \Sigma_{l,n,m}$. DAE systems of the form (1) naturally occur when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer to [1–5] and the references therein. The system $[E, A, B]$ is called *regular*, if the matrix pencil $sE - A$ is regular, that is, $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

The functions $x : \mathbb{R} \to \mathbb{R}^n$ and $u : \mathbb{R} \to \mathbb{R}^m$ are usually called *state* and *input* of the system, resp. However, in the general case, u might be constrained and some of the state variables can play the role of an input. In the present paper we will take the viewpoint of the behavioral approach due to Willems [6], see also [7,8]. Within this framework, the variables of the system do not have the interpretation of states and inputs until an analysis of the system reveals the free variables. These free variables should then be interpreted as inputs, since "they can be viewed as unexplained by the model and imposed on the system by the environment" [9]. This approach obeys the physical meaning of the system variables and it may turn out that in the original model the choice of states and inputs was inappropriate.

The *behavior* of the DAE system (1) is introduced as the following set of solutions of (1):

$$\mathfrak{B}_{[E,A,B]} := \{(x, u) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m) \mid Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l),\ (x, u) \text{ satisfies (1) for almost all } t \in \mathbb{R}\},$$

where $\mathcal{L}^1_{\mathrm{loc}}$ and $\mathcal{AC}$ denote the space of locally (Lebesgue) integrable and absolutely continuous functions, resp. DAE control systems based on the above behavior have been studied in detail e.g. in [1].

Nowadays, the modeling of huge industrial problems and complex physical systems is often performed using automatic modeling tools such as Modelica (https://www.modelica.org/). This approach naturally leads to differential–algebraic systems of the form (1). Since in the automatically generated models it is quite common that redundant equations appear and state and input variables are chosen inappropriately, the system (1) is not regular in general, while the physical background tells that a regular model must exist. Therefore, a remodeling, or a regularization, is often required, see [10].

In the present paper we study the regularization of DAE systems, which relies on a procedure developed in [10] and revisited in [11]. In [10] it is shown that, given any DAE system $[E, A, B] \in \Sigma_{l,n,m}$, by a combination of behavioral equivalence transformation, proportional state feedback and reorganization of variables (due to a possibly inappropriate initial choice of states and inputs) a new system $[E_{\mathrm{reg}}, A_{\mathrm{reg}}, B_{\mathrm{reg}}]$ can be obtained where $sE_{\mathrm{reg}} - A_{\mathrm{reg}}$ is regular and has index at most one. In the linear case, explicit transformations and a characterization of the regularized system have

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been obtained in [12]. In the present paper, we improve the results of [10,12] by showing that an application of state feedback is not necessary. Furthermore, we derive a numerically stable algorithm of cubic complexity which establishes the regularization of the system.

The paper is organized as follows: In Section 2 we introduce some preliminary concepts and notation and give a precise problem formulation. The regularization algorithm, which is the main result of the paper, is presented in Section 3 and proved to be feasible for any given system. Numerical reliability and the computational speed of the regularization algorithm are discussed in Section 4. Section 5 provides a detailed comparison of our algorithm with the method proposed in [10] and in Section 6 we demonstrate the regularization algorithm by means of a numerical example. Conclusions are given in Section 7.

2. Preliminaries and problem formulation

In the present paper we use the following notation: $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, resp.; $\mathbb{R}[s]$ is the ring of polynomials with coefficients in $\mathbb{R}$; $R^{n\times m}$ is the set of $n\times m$ matrices with entries in a ring R; $\mathcal{O}_n$ denotes the set of orthogonal real $n\times n$ matrices. A polynomial matrix $U(s)\in\mathbb{R}[s]^{n\times n}$ is called *unimodular*, if it is invertible over $\mathbb{R}[s]$ or, equivalently, if $\det U(s)$ is a nonzero constant.

The rank of a matrix $M\in\mathbb{K}^{n\times m}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, is denoted by $\operatorname{rk} M$. If $M\in\mathbb{R}^{n\times m}$ with $\operatorname{rk} M=r$, then, using QR factorization with pivoting [13], there exists $T\in\mathcal{O}_n$ such that

$$TM=\begin{bmatrix}\Sigma_r\\0\end{bmatrix},$$

where $\Sigma_r\in\mathbb{R}^{r\times m}$ with $\operatorname{rk}\Sigma_r=r$, see also [14]. We will call T a *row compression* of the matrix M. Similarly, we call $S\in\mathcal{O}_m$ a *column compression*, if

$$MS=[\hat{\Sigma}_r,0],$$

where $\hat{\Sigma}_r\in\mathbb{R}^{n\times r}$ with $\operatorname{rk}\hat{\Sigma}_r=r$.

The index $\nu\in\mathbb{N}_0$ of a regular matrix pencil $sE-A\in\mathbb{R}[s]^{n\times n}$ is defined via its (quasi-)Weierstraß form [15,3,4]: if for some invertible $S,T\in\mathbb{R}^{n\times n}$

$$S(sE-A)T=\begin{bmatrix}sI_r-J & 0\\0 & sN-I_{n-r}\end{bmatrix},\quad N \text{ nilpotent},$$

$$\text{then}\quad \nu:=\begin{cases}0, & \text{if } r=n,\\ \min\left\{\,k\in\mathbb{N}_0\;\middle|\;N^k=0\,\right\}, & \text{if } r<n.\end{cases}$$

The index is independent of the choice of S,T.

Finally, we recall the concept of behavioral equivalence which has been introduced for general behaviors in [9]. Roughly speaking, two systems are behaviorally equivalent, if their behaviors coincide.

Definition 2.1. Two systems $[E_i,A_i,B_i]\in\Sigma_{l,n,m}$, $i=1,2$, are called *behaviorally equivalent*, if

$$\mathfrak{B}_{[E_1,A_1,B_1]}\cap\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m)=\mathfrak{B}_{[E_2,A_2,B_2]}\cap\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m),$$

where $\mathcal{C}^\infty$ denotes the space of infinitely times differentiable functions; we write

$$[E_1,A_1,B_1]\simeq_{\mathfrak{B}}[E_2,A_2,B_2].$$

In order to obtain a behaviorally equivalent system, it is allowed that some of the equations in (1) are differentiated (and hence we require smooth solutions). This leads to a transformation of the form $U(\frac{\mathrm{d}}{\mathrm{d}t})(\frac{\mathrm{d}}{\mathrm{d}t}E-A)x(t)-U(\frac{\mathrm{d}}{\mathrm{d}t})Bu(t)=0$ with some $U(s)\in\mathbb{R}[s]^{l\times l}$. Furthermore, since the behaviors must coincide (on $\mathcal{C}^\infty$) the transformation $U(s)$ must be reversible, i.e., $U(s)$ must be unimodular. As shown in [9, Thms. 2.5.4 & 3.6.2] this is exactly the set of transformations that characterizes behavioral equivalence; this is summarized in the following lemma.

Lemma 2.2. *Let $[E_i,A_i,B_i]\in\Sigma_{l,n,m}$, $i=1,2$. Then $[E_1,A_1,B_1]\simeq_{\mathfrak{B}}[E_2,A_2,B_2]$ if, and only if, there exists a unimodular $U(s)\in\mathbb{R}[s]^{l\times l}$ such that*

$$[sE_1-A_1,-B_1]=U(s)[sE_2-A_2,-B_2].$$

Note that in initial value problems (1), $x(0)=x^0$, where $u\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$ is given, the consistency of the initial value $x^0\in\mathbb{R}^n$, i.e., existence of $x\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n)$ such that $(x,u)\in\mathfrak{B}_{[E,A,B]}$ and $x(0)=x^0$, is preserved under behavioral equivalence.

In the present paper we consider the following regularization problem.

Problem 2.3. For a given system $[E,A,B]\in\Sigma_{l,n,m}$, find a unimodular matrix $U(s)\in\mathbb{R}[s]^{l\times l}$, orthogonal state space and input space transformations $T\in\mathcal{O}_n$, $V\in\mathcal{O}_m$ and a permutation matrix $P\in\mathcal{O}_{n+m}$ such that

$$[sE-A,-B]\begin{bmatrix}T&0\\0&V\end{bmatrix}P=U(s)\begin{bmatrix}0&0\\sE_{\rm reg}-A_{\rm reg}&-B_{\rm reg}\end{bmatrix},\tag{2}$$

where $sE_{\rm reg}-A_{\rm reg}\in\mathbb{R}[s]^{\hat{n}\times\hat{n}}$ is regular and has index at most one.

Each kind of the transformations in Problem 2.3 have an interpretation in terms of their physical meaning:

(i) T and V represent coordinate changes in state space and input space respectively,
(ii) $U(s)$ represents an equivalence transformation which does not change the behavior of the system,
(iii) P represents a permutation of state and input variables. Here, we seek a permutation of free state variables with constraint input variables, so that in the resulting system the free variables are exactly the input variables. This may be viewed as a reinterpretation of certain states as inputs and vice versa.

At first glance it may be surprising that (2) in Problem 2.3 does not read

$$W(s)[sE-A,-B]\begin{bmatrix}T&0\\0&V\end{bmatrix}P=\begin{bmatrix}0&0\\sE_{\rm reg}-A_{\rm reg}&-B_{\rm reg}\end{bmatrix},\tag{3}$$

where $W(s)\in\mathbb{R}[s]^{l\times l}$ is unimodular. The reason is that $U(s)$ in (2) may be easier to compute than $W(s)$ in (3). In fact, we show in Section 3 that $U(s)$ has degree 1, i.e., it is a matrix pencil, and it is obtained with cubic complexity. On the other hand, the inverse $W(s)=U(s)^{-1}$ may have higher degree and can only be obtained with quartic complexity in general, see Section 4.

3. Regularization algorithm

In this section we provide a step by step procedure for the derivation of the regularization of a descriptor system as in (2).

Initialization. Let $[E,A,B]\in\Sigma_{l,n,m}$ be given.

Step 1. Compute a row compression $S_1\in\mathcal{O}_l$ such that $S_1B=\begin{bmatrix}0\\B_2\end{bmatrix}$, where B_2 has full row rank r. Consider

$$S_1[sE-A,-B]=\begin{bmatrix}sE_1-A_1&0\\sE_2-A_2&-B_2\end{bmatrix},$$

where $sE_1-A_1\in\mathbb{R}[s]^{(l-r)\times n}$, $sE_2-A_2\in\mathbb{R}[s]^{r\times n}$.