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Conservation laws and Hamilton's equations for systems with long-range interaction and memory

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Abstract

Using the fact that extremum of variation of generalized action can lead to the fractional dynamics in the case of systems with long-range interaction and long-term memory function, we consider two different applications of the action principle: generalized Noether's theorem and Hamiltonian type equations. In the first case, we derive conservation laws in the form of continuity equations that consist of fractional time–space derivatives. Among applications of these results, we consider a chain of coupled oscillators with a power-wise memory function and power-wise interaction between oscillators. In the second case, we consider an example of fractional differential action 1-form and find the corresponding Hamiltonian type equations from the closed condition of the form.

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1. Introduction

Different physical phenomena such as anomalous transport or random walk with infinite moments [1,2], dynamics of porous media [3,4], continuous time random walk [5–7], chaotic dynamics [8] (see also reviews [9,10]) can be described by equations with fractional integro-differentiation. Despite of fairly deep and comprehensive results in fractional calculus (see [11–14]) a possibility of their applications to physics needs to develop specific physical tools such as extension of fractional calculus to the areas as multi-dimension [11,17], multi-scaling [15,16], variational principles [18,19].

In this paper, we concentrate on two problems important for numerous physical applications: conservation laws and Hamiltonian type equations, both obtained from the corresponding fractional action principles.

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In Section 2, we derive the Noether's theorem for a Lagrangian that includes non-local space–time densities. The Noether's theorem was also discussed in [23,24]. Our new derivation shows in an explicit way how fractional derivative in time emerges from the specific type of the memory function, and how fractional derivative in space is related to a specific long-distance potential of interaction (Section 3.) In Section 4, these results are applied to a chain of nonlinear oscillators that is a subject of great interest in statistics and dynamics [25,26]. Finally, at Section 5, we derive a specific case of fractional Hamilton's equations. Different steps in this direction were performed in [29–31]. We consider the Lagrangian density as a functional without fractional derivatives but, instead, the differential 1-form has fractional differentials. Some examples are given for this type of systems.

The main feature of this paper is the consideration of fractional type differentials or derivatives in both space–time coordinates.

2. Noether's theorem for long-range interaction and memory

2.1. Action and Lagrangian functionals

Let us consider the action functional

$$S[u] = \int_R d^2x \int_R d^2y \mathscr{L}(u(x), u(y), \partial u(x), \partial u(y)), \tag{1}$$

where $x = (t, r)$, t is time, r is coordinate, and $y = (t', r')$, $\partial u(x) = (\partial_t u(t,r), \partial_r u(t,r))$. The integration is carried out over a region R of the two-dimensional space $\mathbb{R}^2$ to which x belong. The field $u(x)$ is defined in the region R of $\mathbb{R}^2$. We assume that $u(x)$ has partial derivatives

$$\partial_0 u(x) = \frac{\partial u(t,r)}{\partial t}, \quad \partial_1 u(x) = \frac{\partial u(t,r)}{\partial r},$$

which are smooth functions with respect to time and coordinate. Here $\mathscr{L}(u(x), u(y), \partial u(x), \partial u(y))$ is generalized density of Lagrangian. If

$$\mathscr{L}(u(x), u(y), \partial u(x), \partial u(y)) = \mathscr{L}(u(x), \partial u(x))\delta(x - y), \tag{2}$$

then we have the usual action functional

$$S[u] = \int_R d^2x \mathscr{L}(u(x), \partial u(x)).$$

The variation of the action (1) is

$$\delta S[u,h] = \int_R d^2x \int_R d^2y \left(\frac{\partial \mathscr{L}}{\partial u(x)} h(x) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu u(x))} \partial_\mu h(x) + \frac{\partial \mathscr{L}}{\partial u(y)} h(y) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu u(y))} \partial_\mu h(y) \right), \tag{3}$$

where $\mu = 0, 1$, $\partial_\mu = \partial/\partial x^\mu$ and

$$\mathscr{L} = \mathscr{L}(u(x), u(y), \partial u(x), \partial u(y))$$

and $h(x) = \delta u(x)$ is the variation of the field u. The variation (3) can be presented as

$$\delta S[u,h] = \int_R d^2x \int_R d^2y \left(\frac{\partial \mathscr{L}_s}{\partial u(x)} h(x) + \frac{\partial \mathscr{L}_s}{\partial(\partial_\mu u(x))} \partial_\mu h(x) \right), \tag{4}$$

where

$$\mathscr{L}_s = \mathscr{L}(u(x), u(y), \partial u(x), \partial u(y)) + \mathscr{L}(u(y), u(x), \partial u(y), \partial u(x)).$$

We can define the functional

$$L[x, u, \partial u] = \frac{1}{2} \int_R d^2y \mathscr{L}_s, \tag{5}$$