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Review

Small-gain stability conditions for linear systems with time-varying delays[☆]

Jing Zhu[a], Tian Qi[b], Jie Chen[a,*]

[a] Department of Electronic Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong Special Administrative Region

[b] Center for Control and Optimization, School of Automation Science and Engineering, South China University of Technology, Guangzhou, China

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ABSTRACT

In this paper we show that a variety of stability conditions, both existing and new, can be derived for linear systems subject to time-varying delays in a unified manner in the form of scaled small-gain conditions. From a robust control perspective, our development seeks to cast the stability problem as one of robust stability analysis, and the resulting stability conditions are also reminiscent of robust stability bounds typically found in robust control theory. The development is built on the well-known conventional robust stability analysis, requiring essentially no more than a straightforward application of the small gain theorem. The derived conditions have conceptual appeal, and they can be checked using standard robust control toolboxes.

Contents

1. Introduction

In this paper we are concerned with linear time-delay systems described by the state-space equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)), \tag{1}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}$ are given constant state matrices, while $\tau(t)$ is the delay parameter varying with time, which satisfies the bounds:

$$0 \le \tau(t) \le \tau_M, \tag{2}$$

$$|\dot{\tau}(t)| \le \delta < 1, \quad \forall t \ge 0. \tag{3}$$

In other words, the time-varying delay is only known to be within a given interval, and how fast it may vary is bounded by a given rate. We study the stability properties of delay systems in this category.

Stability of time-delay systems in the aforementioned description has been long and well studied (see [1–3] and the references therein), and both time- and frequency-domain analysis approaches have been developed. For systems with an unknown constant delay, i.e., when $\tau(t)$ is a constant, the stability problem is largely resolved and various stability conditions are readily available. In particular, necessary and sufficient frequency-domain conditions in the spirit of small-gain theorem can be efficiently computed to determine the stability and characterize other relevant properties for such systems, and more generally for systems with multiple commensurate delays [2,4,5]. On the other hand, the stability problem in the case of time-varying delays proves far more intricate. The existing results are predominantly time-domain conditions, which in their essential flavor are obtained based on the construction of Lyapunov functionals and as the

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* Corresponding author. Tel.: +852 3442 4280.
E-mail addresses: jingzhu3-c@my.cityu.edu.hk (J. Zhu), auqt@scut.edu.cn (T. Qi), jichen@cityu.edu.hk (J. Chen).

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solutions to *linear matrix inequality* (LMI) problems (see [1,2,6] and the references therein).

In this paper we present small-gain type stability criteria for systems subject to time-varying delays. The essential catalyst motivating this development is the classical small-gain theorem governing the stability of interconnected feedback systems. To this effect, a key step is to reformulate delays by fictitious modeling uncertainties in a non-conservative way, which consequently will allow us to draw upon rich tools and techniques in robust control theory. This idea, built upon [7], was first advocated in [2]. Similar results were developed subsequently in, e.g., [8–10], which, albeit not directly in terms of system gains and a small-gain argument, are characterized by quadratic integral inequalities known as *integral quadratic constraints* (IQC) and solvable as LMI problems. This work supplements the existing results by collecting and developing in a unified manner a systematic set of small gain conditions. It is worth pointing out that by their nature some of these conditions are particularly pertinent and indeed comparable to the IQC results alluded to above; in fact, some of them are equivalent and can be derived from one to another. Nevertheless, we hold the perspective that a variety of existing as well as new stability conditions can be derived directly and systematically based on the conventional, more familiar robust stability analysis, in the form of scaled small-gain conditions typically found in, e.g., *structured singular value* analysis [11]. This approach appears both natural and more straightforward, and arguably, renders our results to be of more conceptual transparency and technical simplicity, requiring essentially only the well-known small gain theorem, instead of more sophisticated machineries. Well-established robust control toolboxes can be used directly as well, in a straightforward manner. Additionally, our approach also recovers in the limit the stronger conditions only applicable to constant delays, when the delay variation rate tends to zero, that is, when the delay is very slowly varying. The results in this latter case share the spirit of [12] and do not seem to be available elsewhere.

2. Preliminaries

We denote by $\bar{\sigma}(\cdot)$ the largest singular value of a matrix, and $\rho(\cdot)$ the spectral radius. Let $\|\cdot\|$ be the Hölder ℓ_2 norm of a vector, and L_2 the space of energy-bounded signals defined by

$$L_2 := \{f : \mathbb{R} \to \mathbb{R}^n | \; \|f\|_2 < \infty\},$$

where the L_2 norm $\|\cdot\|_2$ is defined as

$$\|f\|_2 := \left(\int_0^\infty \|f(t)\|^2 \,\mathrm{d}t\right)^{1/2}.$$

Furthermore, let $\mathbb{RH}_\infty$ be the set of stable rational transfer function matrices. Consider a linear system $H : L_2 \to L_2$, so that $y = Hx$. The L_2 induced system norm of H is defined as

$$\|H\|_{2,2} := \sup_{x \neq 0} \frac{\|y\|_2}{\|x\|_2}.$$

The system is said to be L_2-stable if $\|H\|_{2,2}$ is finite. For a linear time-invariant (LTI) system, the L_2 induced system norm coincides with the H_∞ norm of the system's transfer function matrix $\hat{H}(s)$, i.e.,

$$\|\hat{H}\|_\infty = \sup_\omega \bar{\sigma}(\hat{H}(j\omega)).$$

Under the L_2-stability notion, the small-gain theorem can be stated as follows [2, pp. 72], [11, pp. 211]:

Lemma 1 (*Small-Gain Theorem*)**.** *Suppose that M and Δ are both L_2-stable. Then the feedback system in Fig. 1 is L_2-stable if*

$$\|M\Delta\|_{2,2} < 1. \tag{4}$$

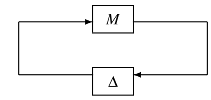

Fig. 1. M–Δ loop: Small-gain theorem.

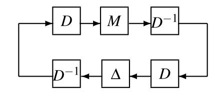

Fig. 2. M–Δ loop: Scaled small-gain theorem.

Furthermore, since $\|M\Delta\|_{2,2} \leq \|M\|_{2,2}\|\Delta\|_{2,2}$, a weaker condition for the system's L_2-stability is

$$\|M\|_{2,2}\|\Delta\|_{2,2} < 1. \tag{5}$$

For convenience, we shall refer to M as the nominal system, Δ as the uncertainty, and the feedback system as the M–Δ loop. It is clear that in general, the small-gain conditions (4) and (5) provide only sufficient stability conditions, which at times can be conservative. The conservatism in the small-gain conditions, nevertheless, can be reduced by introducing matrix multipliers in the feedback loop. Toward this end, we introduce the sets of matrices and transfer function matrices

$$\mathbf{D_S^{(n)}} = \{\mathrm{diag}\{d_1, \ldots, d_n\} : \; d_i \in \mathbb{R}, d_i > 0\},$$
$$\mathbf{D_F^{(N)}} = \{\mathrm{diag}\{D_1, \ldots, D_N\} : \; D_i \in \mathbb{C}^{n \times n}, D_i = D_i^* > 0\},$$
$$\mathbf{D_d^{(N)}} = \{\mathrm{diag}\{D_1(s), \ldots, D_N(s)\} : \; D_i(s), \; D_i^{-1}(s) \in \mathbb{RH}_\infty\},$$

with dimensions compatible to that of Δ. These sets are commonly referred to as constant diagonal, constant block diagonal, and frequency-dependent block diagonal scalings, corresponding to full-block and repeated scalar uncertainties, respectively, in the structured singular value analysis [11]. Noting the equivalence of Fig. 2 to Fig. 1, we conclude that another sufficient condition for the L_2-stability of the M–Δ loop is that

$$\|D^{-1}MD\|_{2,2}\|D^{-1}\Delta D\|_{2,2} < 1, \tag{6}$$

for any D in the aforementioned sets.

As pointed out in [2, pp. 74], the time delay can be considered a linear operator Δ_1 such that

$$\Delta_1 x(t) = x(t - \tau(t)). \tag{7}$$

As such, one may reformulate the system (1) as one of feedback interconnection in Fig. 1; the corresponding M will be given in the next section. This recognition led to a sufficient stability condition in [2], based on the small-gain theorem given in Lemma 1. It is worth noting that for any $D \in \mathbf{D_S^{(n)}}$ and $D \in \mathbf{D_F^{(1)}}$, we have $D\Delta_1 x(t) = \Delta_1 Dx(t)$; in other words, the constant scaling matrix D and the uncertainty Δ_1 commute. The implication then is that $D^{-1}\Delta_1 D = \Delta_1$. According to (5) the M–Δ loop is L_2-stable if the scaled small-gain condition

$$\|D^{-1}MD\|_{2,2}\|\Delta_1\|_{2,2} < 1$$

holds. It is important to emphasize, nonetheless, that for frequency-dependent scaling $D(s) \in \mathbf{D_d^{(1)}}$, this commutability ceases to be true [8], a fact whose importance will be seen shortly.

More refined uncertainty reformulations of the delay can be sought after by employing the so-called *model transformation* [2, pp. 211]

$$x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(u)\,\mathrm{d}u. \tag{8}$$